What is the *characteristic impedance* $Z_0(x, \omega)$ and the *propagation function* $\gamma(x, \omega)$, for inhomogeneous media?

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Abstract

Power flux in transmission lines (TL) is important to understand, and a TL's *characteristic impedance* Z_0 is used in this definition. For uniform TL's, Z_0 is defined in terms of two material properties, $\mathcal{Z}$ and $\mathcal{Y}$, as $P_\pm/U_\pm = \sqrt{\mathcal{Z}/\mathcal{Y}}$, where $[P_\pm, U_\pm]$ are the forward or backward traveling pressure, and velocity, and $[\mathcal{Z}, \mathcal{Y}]$ are the series impedance shunt admittance, per unit length. This fundamental relation has been extend to inhomogeneous TLs, where $\mathcal{Z}(x, s) = sL(x)$, $\mathcal{Y}(x, s) = sC(x)$, and $s = \sigma + i\omega$ is the complex radian frequency. A related extension exists for the *propagation function* $\gamma(x, s) = \sqrt{\mathcal{Z}(x, s)\mathcal{Y}(x, s)}$. These results give insight into energy propagation in inhomogeneous TLs.

0.1 The homogeneous case

The wave equation for the pressure $p(x,t)$ is

$$\frac{\partial^2 p(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p(x,t)}{\partial t^2}. \tag{1}$$

In 1746 Jean-le-Rond D'Alembert (1717-1783) was the first to demonstrate that the general solution of the wave equation is

$$p(x,t) = p_+(t - x/c) + p_-(t + x/c), \tag{2}$$

where $p_+(\xi)$ and $p_-(\xi)$ are two arbitrary functions. This is a most insightful solution, that tells us that the wave equation supports forward and backward propagated "waves," which are represented in D'Alembert's solution as two arbitrary functions p_+ and p_-, traveling at the speed (of sound or light) c, in the forward and backward directions. These two solutions are denoted as *wave variables*, since they are of the form of forward and backward traveling pressure (voltage) or velocity (current) waves. D'Alembert showed that any function,[1] having arguments $t \pm x/c$, is a solution of the wave equation. The incident wave is denoted the *anterograde wave*, while reverse (reflected) traveling wave is denoted the *retrograde wave*.

[1]The differentiability of these functions was a raging debate (Pierce, 1981). Find out D'Alembert's position on the differentiability issues.

$P(x, \omega)$ **waves**

Since the system described is *linear* and *time-shift invariant* (LTI), the solution may be transformed into the *frequency domain*, where the time response is given by $e^{i\omega t}$, such that

$$p(x, t) = P(x, \omega)e^{i\omega t}, \tag{3}$$

which when substituted into Eq. 1, gives

$$\frac{\partial^2 P(x, \omega)}{\partial x^2} + k^2 P(x, \omega) = 0, \tag{4}$$

where $k = \omega/c = 2\pi/\lambda$ is the *wave number* and λ the *wavelength*.

The solution of this frequency domain description of the forward and backward going waves is

$$P(x, \omega) = A_+(\omega)e^{-i\omega x/c} + A_-(\omega)e^{i\omega x/c}, \tag{5}$$

which is simply a Fourier decomposition of forward and backward traveling waves in terms of complex amplitudes $A_+(\omega)$ and $A_-(\omega)$. These amplitudes must be determined by the boundary conditions, which couple the forward and backward traveling waves, since there are no internal reflections.

$\widetilde{P}(k,\omega)$ **waves**

Going one step further via a second Fourier transform (FT) with respect to x, the two plane waves correspond to delta functions in k space. Using the notation of a FT pair $f(x) \Leftrightarrow \widetilde{F}(k)$, denoted by the double headed arrow, the two traveling waves in k space reduce to

$$e^{\mp i\omega x/c} \leftrightarrow \int e^{\mp i\omega x/c} e^{-ikx} dx = 2\pi\delta(k \pm \omega/c). \tag{6}$$

The upper sign corresponds to a wave is traveling to the right, while the lower sign corresponds to a wave to the left. These signs follow from our $e^{i\omega t}$ sign convention. For example, the double inverse FT of $2\pi\delta(k+\omega/c)$ gives a forward traveling, time domain plane wave

$$\delta(t - x/c) = \frac{1}{(2\pi)^2} \int_\omega \int_k 2\pi\delta(k+\omega/c)\; e^{i(kx+\omega t/c)}\; dk\; d\omega. \tag{7}$$

From the decomposition given by Eq. 6, one might conclude that given a mixture of forward and backward traveling waves, it should be possible to separate these waves by "windowing" by a Hilbert transform, the wave number domain into positive and negative k, and then apply an inverse Fourier (or Laplace) transform, $p_\pm(x,t) \leftrightarrow P_\pm(x,\omega)$.to obtain a decomposition of any waves into forward and backward traveling waves.

This might be likened to the procedure developed during WWII by Wiener called *Spectral factorization*, which is closely related to the Hilbert Transform and homomorphic processing Weiner (1945), Bogart and Tukey (1965?).

This decomposition is relevant since we are specifically interested in generalizations of D'Alembert's equation (Eq. 2) to cases of resonant scattering, where $ka \approx 1$. The details on exactly how to do this need elaboration.

Many questions regarding the constitutive relations for wave propagation in inhomogeneous media are not addressed in the literature.

The main question addressed by this report is:
"How does one best characterize and model the power flux in inhomogeneous media?"

We are interested in at least four specific question for the case of inhomogeneous media, widely accepted as true for plane wave propagation in homogeneous media:

- *First*, is it possible to linearly transform pressure and velocity into waves that naturally account for the *forward and backward power transfer*?
- *Second*, is it possible to generalize the characteristic impedance $Z_0(x, s)$ for inhomogeneous media? This question is related to the first.
- *Third*, what is the relation for $\gamma(x, s)$ in terms of the constitutive relations $\mathcal{Z}(x, s)$ and $\mathcal{Y}(x, s)$ for inhomogeneous media? This question is related to the second.
- *Fourth*, can we take these relations into the tme domain, as defined in terms of time-domain convolutions?

These questions seem to be related to *Huygen's principle* (c1678), which is a key concept in wave propagaion (Christian Huygens, 1629-1695):

> *The wavefront of a propagating wave of light at any instant conforms to the envelope of spherical wavelets emanating from every point on the wavefront at the prior instant.*

This principle is deficient in that it fails to account for the directionality of the wave propagation in time.

Energy flow (power) has direction, collinear with that of the velocity. The total pressure and (volume) velocity have corresponding forward and retrograde components. It follows from wave linearity, that without loss of generality, the pressure (a force per unit area) may be written as a sum of anterograde P^+ and retrograde P^- propagated pressure waves as

$$P(x, \omega) = P^+(x, \omega) + P^-(x, \omega). \tag{8}$$

Likewise, the axial velocity U (a flow) may be expressed as a "sum" of anterograde and retrograde axial velocity traveling wave components

$$U(x, \omega) = U^+(x, \omega) - U^-(x, \omega). \tag{9}$$

The retrograde velocity component is negative because velocity, like power, has direction, whereas pressure is a scalar, without direction.

We would like to show that the ratio of the pressure and velocity for forward and backward traveling waves is equal to an *inhomogeneous impedance* we call the *characteristic impedance*, denoted $\zeta_0(x,t)$ in the time domain and $Z_0(x,s)$ in the frequency domain, such that

$$\zeta_0(x,t) \leftrightarrow Z_0(x,s) = \frac{P_\pm(x,\omega)}{U_\pm(x,\omega)}, \quad \text{[Ohm's Law]} \tag{10}$$

independent of the direction of the wave (as above).

For homogeneous waves Eq. 10 is well known as defined in Pipes (1958, Page 293).

The situation for inhomogeneous waves, where $\mathcal{Z}(x,s)$ and $\mathcal{Y}(x,s)$ depend on position, is much less clear.

How do we uniquely define the waves?

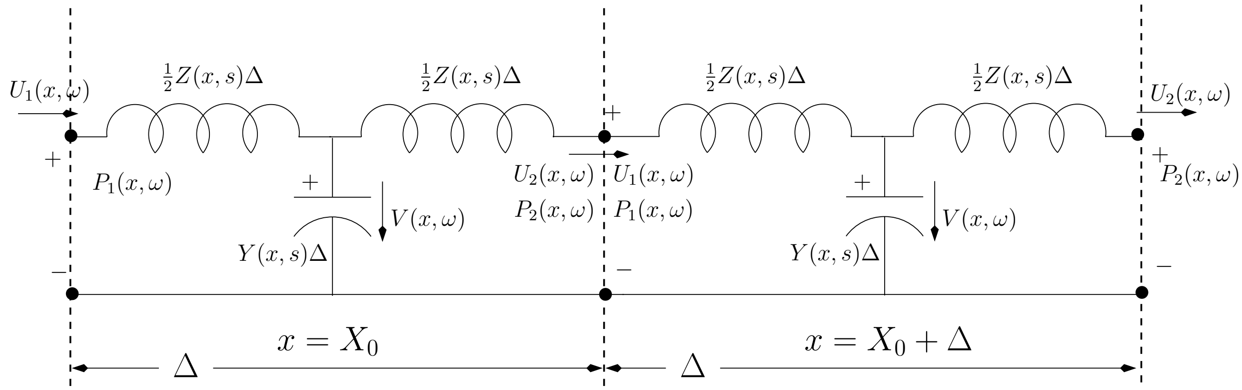


Figure 1: *Two sections of the basic transmission line model, as a lumped parameter electrical circuit. Inductance represents mass per unit length and while the capacitance represents stiffness per unit length, in such a representation, for a section Δ meters long.*

A medium's wave properties are characterized by the *per-unit length series impedance* $\mathcal{Z}(x,s)$ and the *per-unit length shunt admittance* $\mathcal{Y}(x,s)$, denoted here as *constitutive relations*, where $s = \jmath\omega$ is the complex Laplace (radian) frequency.

Wave propagation depends on the physics of these constitutive relations since they characterize the medium.

The functions $\mathcal{Z}$ and $\mathcal{Y}$ are determined solely by experimental and modeling questions. We are interested in cases where these functions are *inhomogeneous* (function of position x) and *dispersive* (functions of complex radian frequency $s = \sigma + i\omega$). However the mathematical properties of inhomogeneous media are the primary topic of this report, with a minor theme being dispersion and causality. In the following we shall limit our discussion to one-dimensional wave propagation.

We shall show, that just as in the homogeneous case, the two inhomogeneous material properties $\mathcal{Z}(x,s)$ and $\mathcal{Y}(x,s)$ define the wave properties via two functions, the *wave characteristic impedance*,

$$Z_0(x,s) \equiv \sqrt{\frac{\mathcal{Z}(x,s)}{\mathcal{Y}(x,s)}}, \tag{11}$$

and the *wave propagation function*

$$\gamma(x,s) = \sqrt{\mathcal{Z}(x,s)\mathcal{Y}(x,s)}. \tag{12}$$

The wave characteristic impedance is important because it is used in the definition of the *wave power*, while the wave propagation function determines the *wave velocity*, as a function of position and frequency, allowing for the generalize of D'Alembert's solution of the wave equation (see Eq. 2 below) to the case of inhomogeneous and dispersive media.

Given *Ohm's Law* (Eq. 10), it follows that the average forward (+) and retrograde (−) power is

$$\overline{\mathcal{P}}_\pm(x,\omega) \equiv \frac{1}{2}\Re P_\pm U_\pm^* = \frac{1}{2}\Re Z_0|U_\pm(x,\omega)|^2 = \frac{1}{2}\Re Y_0(x,s)|P_\pm(x,\omega)|^2. \tag{13}$$

where $Y_0 = 1/Z_0$ and $\Re$ indicates the real part.

For example: for the case of Maxwell's equations, μ is the free-space inducance per unit length, ϵ is the free-space capacitance per unit length, and $\mathcal{Z} = s\mu$ and $\mathcal{Y} = s\epsilon$. Thus the formual for the *characteristic impedance* of free space is

$$Z_0 \equiv \sqrt{\frac{\mathcal{Z}}{\mathcal{Y}}} = \sqrt{\mu/\epsilon} = 377 \quad [\Omega]. \tag{14}$$

Likewise the propagation function is

$$\gamma \equiv \sqrt{\mathcal{Z}\mathcal{Y}} = s\sqrt{\mu\epsilon}. \tag{15}$$

Since $\gamma = j\omega/c$, the speed of light is

$$c \equiv 1/\sqrt{\mu\epsilon} = 3 \times 10^9 \quad [\text{m/s}]. \tag{16}$$

For the *acoustics case*, the series mass impedance per unit length of air is

$$\mathcal{Z} = s\rho_0 \tag{17}$$

and the admittance per unit length of air is

$$\mathcal{Y} = s/\gamma_0 P_0. \tag{18}$$

For the case of sound in air, $\gamma_0 \equiv c_p/c_v = 1.4$ is the ratio of specific heats and $P_0 = 10^5$ [Pa] (mks-units) is the ambient static pressure. Thus the *specific acoustic impedance* of air is

$$Z_0 \equiv \sqrt{\frac{\mathcal{Z}}{\mathcal{Y}}} = \sqrt{\rho_0 \gamma P_0} = \rho_0 c \;\; [\Omega] \tag{19}$$

and the propagation function is

$$\gamma \equiv \sqrt{\mathcal{Z}\mathcal{Y}} = s\sqrt{\frac{\rho_0}{\gamma_0 P_0}} \tag{20}$$

giving the speed of sound as

$$c \equiv \sqrt{\frac{\gamma_0 P_0}{\rho_0}}. \tag{21}$$

Historical definition of $Z_0(x,s)$: Historically the characteristic impedance defined as the input impedance of a semi-infinite line is an alternative definition of $Z_0(x,s)$, which depends on the boundary condition at infinity, and has an impulse response that explores the entire length of the line, to $x \to \infty$ (Schelkunoff, 1943). This definition makes no sense as it is a nonlocal definition.

The characteristic impedance plays the role of the Thévenin impedance described above, and based on this causality argument, it must be strictly local.

If power is to flow locally in going from time t to $t + \delta t$, and if the power traveling in different directions are to be independent of each other, then the characteristic impedance must be local and real.

Can we make it work?

The pressure and velocity are related by the *line impedance* $Z(x, s)$, defined as the pressure over the velocity

$$\zeta(x,t) \leftrightarrow Z(x,s) \equiv \frac{P(x,\omega)}{U(x,\omega)} = \frac{P^+ + P^-}{U^+ - U^-} = Z_0(x,s)\frac{1+\mathcal{R}(x,s)}{1-\mathcal{R}(x,s)}, \tag{22}$$

where we have used our previous definitions Eq. 8, Eq. 9 and Eq. 10.

In the time domain $p(x,t) = \zeta(x,t) \star u(x,t)$ at every point x on the line, where $\star$ represents the time convolution operator, namely $p_-(x,t) = \int_{\tau=0^-}^{\infty} \varrho(x,\tau)p_+(x,t-\tau)d\tau$.

In Eq. 22 the *reflectance* $\mathcal{R}(x,s)$ is defined in the frequency domain as the transfer function between incident and reflected waves at a location x along the line, namely

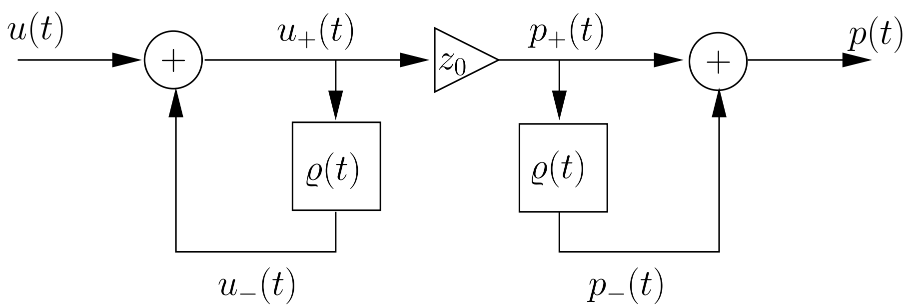
$$\varrho(x,t) \leftrightarrow \mathcal{R}(x,s) \equiv \frac{P_-(x,\omega)}{P_+(x,\omega)} = \frac{U_-(x,\omega)}{U_+(x,\omega)}. \tag{23}$$

The wave variables $P_\pm, U_\pm$ are related to each other via the reflectance. For example, in the time domain $p_-(x,t) = \varrho(x,t) \star p_+(x,t)$. A similar relation holds the velocity wave variables.

From the above definition of the reflectance, and from Eq. 22, a little algebra gives

$$\mathcal{R}(x,s) = \frac{\widetilde{Z}(x,s) - 1}{\widetilde{Z}(x,s) + 1}, \tag{24}$$

where $\widetilde{Z}(x,s) \equiv Z(x,s)/Z_0(x,s)$ defines the *normalized impedance*. Given $\widetilde{Z}(x,s)$, one may uniquely find the reflectance $\mathcal{R}(x,s)$ from Eq. 24. Alternatively, given $\mathcal{R}(x,s)$ one may uniquely find the normalized impedance $\widetilde{Z}(x,s)$. Physically this shows that the reflectance is mathematically related to the impedance, normalized via Z_0. This proves that the characteristic impedance for inhomogeneous and dispersive media plays a fundamental role in relating wave variables to the wave power.



A signal-flow graph represention of a 1-port impedance network, that converts velocity to pressure, via the reflectance. This network first computes $u_+(t) = \varrho(t) \star u_+(t) + u(t)$ *given* $u(t)$. *Then* $p_+(t) = z_0 u_+(t)$. *Finally the pressure* $p(t) = p_+(t) + \varrho(t) \star p_+(t)$. *For computability reasons it is necessary that* $\varrho(t \leq 0) = 0$, *namely that the reflectance be strictly causal. The* surge impedance *is defined as the impedance at* $t = 0$. *The product of the input and output gives the total power.*

Basic 1D equation for dispersive inhomogeneous media

An important and well known generalization of Eq. 4 is the *LTI, causal, dispersive, inhomogeneous* pair of first order equations

$$\frac{d}{dx}\begin{bmatrix} P(x,\omega) \\ U(x,\omega) \end{bmatrix} = -\begin{bmatrix} 0 & \mathcal{Z}(x,s) \\ \mathcal{Y}(x,s) & 0 \end{bmatrix}\begin{bmatrix} P(x,\omega) \\ U(x,\omega) \end{bmatrix}. \tag{25}$$

where $P(x,\omega)$ is the pressure (or voltage), $U(x,\omega)$ is the *velocity* (or current),

$$\mathcal{Z}(x,s) = R(x,s) + sL(x,s), \tag{26}$$

is the per unit length series impedance, and

$$\mathcal{Y}(x,s) = G(x,s) + sC(x,s) \tag{27}$$

is the per unit length shunt admittance, where by definition R and G are the real part of $\mathcal{Z}$ and $\mathcal{Y}$, and in a loss-less medium are zero.

This equation is more general than the wave equation Eq. 1. For example, one cannot determine the formula for the wave speed c in terms of $\mathcal{Z}$ and $\mathcal{Y}$ without an additional relationship. However given Eq. 25, this relation may easily be determined, since $\mathcal{Z}$ and $\mathcal{Y}$ are explicitly specified in the formulation.

Equation 25 is called the *Webster Horn equation,* (in acoustic applications) (Webster, 1919), or the *Telegraph equation* (in electromagnetic applications) (Slater, 1942, Pages 69-78).

Equation 25 it is typically written as a single second-order equation.

Wave Variables

We proceed by assuming expressions in terms of wave variables and then evaluate the Horn Equation. From the definition of power, given by Eq. 33, and the separation of the pressure and velocity into wave variables

$$\begin{bmatrix} P(x,\omega) \\ U(x,\omega) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ Y_0 & -Y_0 \end{bmatrix} \begin{bmatrix} P^+(x,\omega) \\ P^-(x,\omega) \end{bmatrix}, \tag{28}$$

where

$$Y_0(x,s) = \frac{1}{Z_0(x,s)} = \sqrt{\frac{\mathcal{Y}}{\mathcal{Z}}} \tag{29}$$

Equation 28 follows from our key requirement

$$Z_0(x,s) \equiv \sqrt{\frac{\mathcal{Z}(x,s)}{\mathcal{Y}(x,s)}} = \frac{P_\pm(x,\omega)}{U_\pm(x,\omega)}. \tag{30}$$

Webster's equation for $P^{\pm}(x,\omega)$

Substitution of Eq. 28 into Eq. 25, and defining $Y_0' \equiv \frac{d}{dx}Y_0(x,\omega)$, renders

$$\frac{d}{dx}\begin{bmatrix} P^+(x,\omega) \\ P^-(x,\omega) \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} Y_0\mathcal{Z} + Z_0(\mathcal{Y}+Y_0') & -Y_0\mathcal{Z} + Z_0(\mathcal{Y}-Y_0') \\ Y_0\mathcal{Z} - Z_0(\mathcal{Y}+Y_0') & -Y_0\mathcal{Z} - Z_0(\mathcal{Y}-Y_0') \end{bmatrix}\begin{bmatrix} P^+(x,\omega) \\ P^-(x,\omega) \end{bmatrix}.$$

From the definitions of Z_0 and $\gamma(x,s)$

$$\gamma(x,s) = Z_0(x,s)\mathcal{Y}(x,s) = Y_0(x,s)\mathcal{Z}(x,s),$$

and defining

$$\varepsilon(x,s) = \frac{1}{2}Z_0Y_0' = \frac{d}{dx}ln(\sqrt{Y_0}), \tag{31}$$

the above equation may be greatly simplified:

$$\frac{d}{dx}\begin{bmatrix} P^+(x,\omega) \\ P^-(x,\omega) \end{bmatrix} = \begin{bmatrix} -\gamma-\varepsilon & \varepsilon \\ \varepsilon & \gamma-\varepsilon \end{bmatrix}\begin{bmatrix} P^+(x,\omega) \\ P^-(x,\omega) \end{bmatrix}. \tag{32}$$

If one ignores the coupling, integration of the upper equation gives the "WKB solution."

We denote this resulting transformed Webster equation (Eq. 25) the *wave evolution equation* (WEE) which has transformed the solution of Eq. 25 in terms of pressure wave variables $P^+(x,\omega)$ and $P^-(x,\omega)$. Eq. 32 has an exclusive relation to power propagation

$$\overline{\mathcal{P}}_{\pm}(x,\omega) \equiv \frac{1}{2}\Re P_{\pm}U_{\pm}^* = \frac{1}{2}\Re Z_0|U_{\pm}(x,\omega)|^2 = \frac{1}{2}\Re Y_0(x,s)|P_{\pm}(x,\omega)|^2. \tag{33}$$

Power and wave variables

In the homogeneous case the forward and backward wave solutions of Eq. 25 are independent, and obey Ohm's Law Eq. 10. For the inhomogeneous case the solutions of Eq. 25 are coupled, due to the back scattering associated with the reflections that occur. However one may still define and identify the forward and backward traveling power by requiring Ohm's Law to be simultaneously obeyed, proportioning the solutions for the inhomogeneous case along the lines of power transfer. This is the role of Eq. 32 via transformation Eq. 28, subject to Eq. 30.

Working in the time domain, the instantaneous power $\mathcal{P}(x,t)$, at every location x and time t, may be expressed in terms of wave variables

$$\mathcal{P}(x,t) \equiv p(x,t)u(x,t) = [p^+(x,t) + p^-(x,t)][u^+(x,t) - u^-(x,t)]. \tag{34}$$

Expanding this gives the total power (at every time instant t and every location x) in terms of these wave variables

$$\mathcal{P}(x,t) = p^+(x,t)u^+(x,t) - p^-(x,t)u^-(x,t) + [p^-(x,t)u^+(x,t) - p^+(x,t)u^-(x,t)]. \tag{35}$$

The first term on the right is denoted the forward traveling (*anterograde*) power

$$\mathcal{P}^+(x,t) = p^+(x,t)u^+(x,t), \tag{36}$$

while the second

$$\mathcal{P}^-(x,t) = p^-(x,t)u^-(x,t) \tag{37}$$

is denoted the reflected (*retrograde*) power. This leaves the third term in square brackets, which we denote the *cross-power*

$$\mathcal{P}_c(x,t) \equiv [p^-(x,t)u^+(x,t) - p^+(x,t)u^-(x,t)]. \tag{38}$$

Lossless homogeneous case: Eq. 10 requires that the cross-power is zero, because

$$Z_0 \equiv \sqrt{\frac{L}{C}} = \frac{P^+}{U^+} = \frac{P^-}{U^-} \tag{39}$$

is a constant. Transforming Eq. 39 to the time domain results in $p_\pm = Z_0 u_\pm$. Substitution into the cross-power and factoring out the common constant Z_0 results in

$$\mathcal{P}_c(x,t) = Z_0[u^-(x,t)u^+(x,t) - u^+(x,t)u^-(x,t)] = 0. \tag{40}$$

In conclusion, for the lossless homogeneous system

$$\mathcal{P}(x,t) = \mathcal{P}^+(x,t) - \mathcal{P}^-(x,t), \tag{41}$$

namely the power absorbed is the forward traveling power less the backward traveling power. This physically makes sense in the sense that the power either is traveling one way or the other (homogeneous property), since no energy is burned up in the network (lossless property). The cross power is always zero since there is no fixed relationship between waves going in the two directions.

Approximate solutions

The WEE (Eq. 32) has a simple approximate solution for $P^+(x,\omega)$, if and when we may ignore the back-scattered wave (Brekhovskikh, 1980; Durney and Johnson, 1969). Under such a condition we may find an approximate solution for $P^-(x,\omega)$ in terms of this approximate $P^+(x,\omega)$, and this leads to a Born-like iterative procedure for $P^+(x,\omega)$.

In cases when $|\varepsilon P^-| << |(\gamma+\varepsilon)P^+|$, the uncoupled forward solution is given by the integration of

$$\frac{d}{dx}P_1^+(x,\omega) \approx -[\gamma(x,s)+\varepsilon(x,s)]P_1^+(x,\omega), \tag{42}$$

resulting in

$$P_1^+(x,\omega) = P^+(0,f)e^{-\int_0^x[\gamma(\xi,\omega)+\varepsilon(\xi,\omega)]d\xi}. \tag{43}$$

This is precisely the well known "WKB approximation" (Schiff, 1955; LePage, 1961).

As shown in the Appendix, integration of the lower of Eq. 32 with the above expression for $P^+(x,\omega)$ as the input leads to the formula for $P_1^-(x,\omega)$

$$P_1^-(x,\omega) = e^{\int_0^x[\gamma(u)-\varepsilon(u,\omega)]du} \int_L^x \varepsilon(\xi,\omega)P_1^+(\xi)e^{-\int_0^\xi[\gamma(u)-\varepsilon(u,\omega)]du}d\xi. \tag{44}$$

The RHS of this equation is the driving term which is a product of the forward going wave $P_1^+(x,\omega)$, the scattering term $\varepsilon(x,s)$ due to the inhomogeneity, and a delay term that accounts for propagation delay and the loss due to the scattering. If we knew the exact value for $P^+(x,\omega)$ rather than the approximate value represented by $P_1^+(x,\omega)$, the equation for $P^-(x,\omega)$ would be an exact relation. It is approximate only in that P_1^+ is approximate.

It seems that this relation deals with the *turning point problem* and predicts *stationary states* (evanescent waves).

Example of the exponential horn

We use the notation and formulation by Leach (1996) for the horn having area $A(x) = A_0 e^{2mx}$, $\mathcal{Z}(x,s) = \rho_0 s/A(x)$ and $\mathcal{Y}(x,s) = sA(x)/\gamma_0 P_0$. Thus $\gamma(s) \equiv \sqrt{\mathcal{Z}\mathcal{Y}} = s/c$ is independent of x, corresponding to dispersionless wave propagation, while $Z_0(x) = \sqrt{\mathcal{Z}/\mathcal{Y}} = \rho_0 c/A(x)$ is independent of frequency (e.g., dispersionless) (Leach, 1996).

Pressure waves satisfying the Webster Equation (Eq. 25) are excited by a pressure source at $x = 0$, leading to

$$P(x,s) = \left[\frac{A_0}{A(x)}\right]^{\frac{1}{2}} \left[P_r(\omega)e^{-\bar{\gamma}x} + P_l(\omega)e^{\bar{\gamma}x}\right], \tag{45}$$

where $P_r(\omega)$ and $P_l(\omega)$ represent the general case of *independent pressure sources* for *right* and *left* traveling waves, (the same as A_+ and A_- of Eq. 5).

Substitution of Eq. 45 into Eq. 25 gives the *dispersion relation*

$$\bar{\gamma}(s) \equiv \pm\sqrt{(s/c)^2 + m^2}, \tag{46}$$

which indicates that wave propagation ceases for frequencies below the *cutoff frequency* $\omega_c = mc$, where the wave number $\bar{k}(s) = \Im\bar{\gamma}(s)$ becomes zero ($\lambda \to 0$). This relation is show in Fig. 2, as radian frequency $\omega(k)$ verses wave number $k = 2\pi/\lambda$. Energy decays as it propagates, but is not lost, rather it is stored as kinetic energy. Above the cutoff frequency, propagating dispersive waves are seen, and grow (or decay) as $[A_0/A(x)]^{0.5} \equiv e^{-mx}$, giving the horn its impedance matching properties.

The velocity wave corresponding to Eq. 45 is (Leach, 1996)

$$U(x,s) = \left[\frac{A_0}{A(x)}\right]^{\frac{1}{2}} \left[Y_r(x,s)P_r(\omega)e^{-\bar{\gamma}x} - Y_l(x,s)P_l(\omega)e^{\bar{\gamma}x}\right]. \tag{47}$$

The determination of P_r and P_l requires *boundary conditions*, which depend on the physical connections (Leach, 1996). As defined, Eq. 47 only applies to a finite horn. When either P_l or P_r are set to zero, Eq. 45 describes the right or left launched wave, into a semi-infinite horn. The

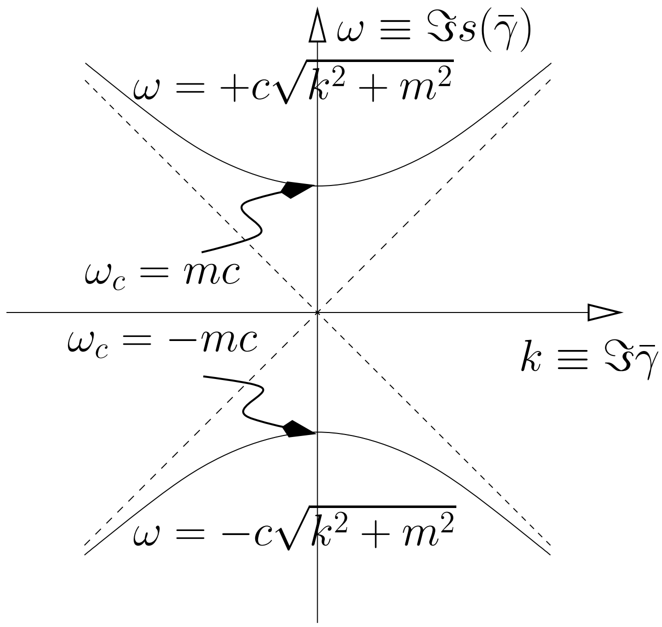


Figure 2: *This shows the relation between the frequency and the wave number k for the exponential horn.*

relevant boundary conditions for this case are the pressure at $x = 0$ and the Sommerfeld boundary condition, at the limit points (Pierce, 1981, Page 362).[2]

Given these modified boundary conditions, the physical meaning of the various terms becomes clear. $Y_r(0, s)$ is the input impedance of the semi-infinite horn looking to the right, while $Y_l(0, s)$ is the same looking to the left. Thus we split the horn into two semi-horns, having a total current

[2]Pierce says "If one disregards the inapplicability of the Webster horn equation at large L . . . ." In fact there is no reason I know of to consider this equation inapplicable. Rather it is the solution Eq. 45 this is in trouble, since it only applies over a finite rage of x. It specifically does not apply to the semi-infinite horn, which requires the application of the Sommerfeld radiation condition at each end.

that is the sum of right and left. The semi-horn input admittances Y_r and Y_l are defined as ratio of the volume velocity over the applied pressure, such that $Y_r(0,s) \equiv U_r(0,\omega)/P_r(\omega)$ and $Y_l(0,s) \equiv U_l(0,\omega)/P_l(\omega)$. If only one semi-horn is present, one of the currents is zero.

Following Leach (1996), $Y_r(x,s)$ and $Y_l(x,s)$ are found to be

$$Y_{r,l}(x,s) = \frac{Y_0(x)}{s}\left[\sqrt{s^2+c^2m^2} \pm 2cm\right]. \tag{48}$$

These two "wave" admittances obviously differ for the two directions, unlike the characteristic admittance $Y_0(x,s)$. The reason for this difference is "non-local" resonant scattering, with one tube increasing and the other decreasing in area. If the line is modified at some point beyond the measurement point, the corresponding admittance must change. Each admittance is the sum of a mass term, and a component having a branch cut between $\omega = \pm mc$. At high frequencies the admittances approach Y_0. What looks like a pole at $s = 0$, is removed by the branch cut. At low frequencies, the wave admittance goes to 0 looking in the decreasing direction, and is mass $2cmY_0/s$, looking into the increasing end of the horn. Further analysis of this admittance would be instructive, but is outside the scope of the present discussion.

Consistent with (Schelkunoff, 1943), Leach calls these half-horn input admittances Y_r and Y_l "characteristic admittances." For the nonuniform line, these two input admittances are not equal, and are clearly distinct from *characteristic admittance* $Y_0(x,s) \equiv 1/Z_0(x,s)$ define via Eq. 11.

While the exact input admittance is given by Eq. 48, the input impedance for each semi-horn may be approximated by a mass in parallel with a resistance (Salmon, 1946a). This resistance does not represent an energy loss, but accounts for the flow of kinetic energy into the reactive trap, since the horn is lossless.

Impulse response of the horn: From tables of inverse Laplace transforms, causality may be verified since

$$\Phi(x,s) \equiv e^{-\bar{\gamma}x} = e^{-\sqrt{s^2+c^2m^2}x/c} \tag{49}$$

is the Laplace transform of

$$\phi(x,t) = \delta(t-x/c) - |m|\frac{x}{c}\frac{J_1\left(|m|c\sqrt{t^2-(x/c)^2}\right)}{\sqrt{t^2-(x/c)^2}}\Delta(t-x/c) \tag{50}$$

where $\delta(t)$ is the impulse function, and $\Delta(t)$ is the Heaviside step function [$\delta(t) = d\Delta(t)/dt$]. From Eq. 45

$$p_r(x,t) = e^{-mx}\phi(x,t). \tag{51}$$

namely $p_r(x,t) = e^{-mx}\mathcal{L}^{-1}[e^{-\bar{\gamma}x}]$.

From the time response we see that the horn transfer function consists of the sum of a delayed wideband pulse (direct sound) and a delayed dispersive low frequency part, represented by the $J_1(t)\Delta(t)/t$ Bessel function.

It can be helpful to think of this as “resonant scattering,” where the wavelength becomes similar to $\lambda \approx 2\pi/m$, leading to a cutoff frequency of $\omega_c = mc$. This is easily understood from the underlying physics of the problem, and is due to internal reflections caused by “rapid” changes in area at sufficiently long wavelengths, namely when $k/m << 1$, which is similar to $ka << 1$ seen in resonant scattering and diffraction problems, with $1/m$ playing the role of the radius a (Schiff, 1955). This similarity is not coincidental.

Schelkunoff's (and thus Leach's) definition of the "characteristic impedance" is $Z_r = 1/Y_r$ and $Z_l = 1/Y_l$. In this case the right and left launched waves (i.e., $[P_r, U_r]$ and $[P_l, U_l]$) do *not* describe uncoupled power waves. For example, if a load admittance is $\sqrt{\mathcal{Y}/\mathcal{Z}}$ and real, all the power delivered by wave variables (e.g., a current source U_r) is delivered to the forward traveling wave, since the impedance has no memory (all the power is delivered to the wave and never returns to the port). If a pressure or current source is applied to $Y_r(s, x)$ or $Y_l(s, x)$, power is returned to the input, since these impedances depend on frequency, and thus have an impulse response that is not a delta function.

Conclusions

1. Right or left propagated waves, and wave variables, are distinct for inhomogeneous systems.

2. Wave variables determine the power flow for inhomogeneous systems.

3. The anterograde and retrograde power is given by the product of the wave variables. Example: The instantaneous retrograde power is given by p_-u_-, while the average (RMS) anterograde power is $\frac{1}{2}\Re P_+U_+^*$.

4. The characteristic impedance is given by the ratio of the wave variable P_+ over U_+ (i.e., Ohms Law), even for inhomogeneous waves Eq. 10, once the distinction between a right launched wave P_r and the wave variable P_+ has been made.

5. The wave impedance Eq. 10 is quite distinct from $Z_0(x, s)$, and it depends on direction ($z_l \neq z_r$). at least for the cases we have studied.

6. The *reflectance* is given by the ratio of the wave variables in pressure or velocity (voltage or current) (Eq. 24), which like Z_0 does not depend on direction (for isotropic media).

7. The WKB approximation massively fails in the cutoff regions $\omega < \omega_c = mc$.

8. Out of the cut-off region ($\omega > mc$ for the exponential horn), the system of equations is inherently first order (e.g., first order solutions, such as the WKB solution make excellent approximations), weakly coupled.

9. In the cutoff region ($\omega < mc$ for the exponential horn), the system of equations is inherently second order (first order approximations necessarily fail).

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I would like to thank Steve Neely and Chris Shera for critical discussions, and my student Feipeng Li for helpful calculations regarding the horn.

Appendix I

The proof is based on the chain rule, as follows:

$$\frac{d}{dx}\left[P(x)e^{-\int_0^x[\gamma(\xi)-\varepsilon(\xi)]d\xi}\right] = e^{-\int_0^x[\gamma(\xi)-\varepsilon(\xi)]d\xi}\frac{dP(x)}{dx} + P(x)\frac{d}{dx}e^{-\int_0^x[\gamma(\xi)-\varepsilon(\xi)]d\xi}$$

Evaluating the final derivative in the above expansion gives

$$-e^{-\int_0^x[\gamma(\xi)-\varepsilon(\xi)]d\xi}\frac{d}{dx}\int_0^x[\gamma(\xi)-\varepsilon(\xi)]d\xi = -[\gamma(x)-\varepsilon(x)]e^{-\int_0^x[\gamma(\xi)-\varepsilon(\xi)]d\xi}.$$

Putting these together, and moving the $e^{-\int \cdots dx}$ to the LHS, gives the "integration by parts" formula required to integrate Eq. 52

$$\left[e^{\int_0^x[\gamma(\xi)-\varepsilon(\xi)]d\xi}\right]\frac{d}{dx}\left[e^{-\int_0^x[\gamma(\xi)-\varepsilon(\xi)]d\xi}P(x)\right] = \frac{dP(x)}{dx} - [\gamma(x)-\varepsilon(x)]P(x).$$

The equation for the retrograde wave $P^-(x,\omega)$ is given by the lower equation of Eq. 32, in terms of $P^+(x,\omega)$

$$\frac{d}{dx}P_1^-(x,\omega) - (\gamma-\varepsilon)P_1^-(x,\omega) = \varepsilon(x,\omega)P_1^+(x,\omega), \tag{52}$$

where the previous equation provides the approximate estimate of $P_1^+(x,\omega)$, which has been placed on the right. Once given $P_1^+(x,\omega)$, Eq. 52 may be rewritten as

$$\frac{d}{dx}\left[e^{-\int_0^x[\gamma(\xi)-\varepsilon(\xi)]d\xi}P_1^-(x,\omega)\right] = \varepsilon(x)P_1^+(x,\omega)\ e^{-\int_0^x[\gamma(\xi)-\varepsilon(\xi)]d\xi}. \tag{53}$$

$$P_1^-(\zeta,f)\ e^{-\int_0^\zeta[\gamma(u)-\varepsilon(u)]du}\Big|_{\zeta=L}^{x} = \int_L^x \varepsilon(\xi)P_1^+(\xi)e^{-\int_0^\xi[\gamma(u)-\varepsilon(u)]du}d\xi + P(x=L). \tag{54}$$

For this final evaluation we assumed that the pressure at $x = L$ is zero, which provides the first order expression for the retrograde wave $P_1^-(x,s)$ in terms of the first-order solution for the forward wave $P_1^+(x,s)$

$$P_1^-(x,\omega) = e^{\int_0^x [\gamma(u)-\varepsilon(u)]du} \int_L^x \varepsilon(\xi) P_1^+(\xi) e^{-\int_0^\xi [\gamma(u)-\varepsilon(u)]du} d\xi. \tag{55}$$

This equation may be further simplified due to the definition of ε Eq. 31, since

$$e^{\int_L^x \varepsilon(u)du} = e^{\ln(\sqrt{Y_0(x)/Y_0(L)})} = \sqrt{Y_0(x)/Y_0(L)}. \tag{56}$$

$$\left[e^{\int_0^x [\gamma(\xi)-\varepsilon(\xi)]d\xi}\right] \frac{d}{dx}\left[e^{-\int_0^x [\gamma(\xi)-\varepsilon(\xi)]d\xi} P(x)\right] = \frac{dP(x)}{dx} - [\gamma(x)-\varepsilon(x)]P(x).$$

Appendix II

May 3, 2007, Steven Elliott has proposed that I define the characteristic impedance by assuming that the cross power is zero. This is a very reasonable proposal that I had considered, but did not see to the end. Steven has followed this reasoning throught, and here is a summary of his argument, and its conclusion:

Twice the total average power is given in the frequency domain as

$$2\mathcal{P} \equiv \Re PU^* = \Re(P_+ + P_-)(U_+^* - U_-^*) = \Re(P_+U_+^*) - \Re(P_-U_-^* + \Re(P_-U_+^* - P_+U_-^*).$$

Each of the three terms in () defines a corresponding power term for the forward power, retrograde power and cross power.

We assume that the forward and backward traveling waves each have a different characteristic impedance

$$Z_+ \equiv \frac{P_+}{U_+}$$

and

$$Z_- \equiv \frac{P_-}{U_-}.$$

In this case the cross power may be written

$$2\mathcal{P}_x = \Re(Z_-U_-U_+^* - Z_+U_+U_-^*).$$

The key part of the argument is that the forward and retrograde waves are, in general, independent. If the cross power is to be zero, it follows that

$$Z_- = Z_+^*.$$

For this to be true the characteristic impedance must be real. The conjugate operator corresponds to time reversal, and if the impedance is to be causal, it can only be real. This argument need ther study, I think.

The conclusion is that if the cross power is to be zero, the characteristic impedance must be a real.

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