Advection-Mediated Coexistence Of Competing Species

Robert Stephen Cantrell
Department of Mathematics, University of Miami
Coral Gables, FL 33124, USA

Chris Cosner
Department of Mathematics, University of Miami
Coral Gables, FL 33124, USA

Yuan Lou
Department of Mathematics, The Ohio State University
Columbus, OH 43210, USA

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Abstract. We study a Lotka-Volterra reaction-diffusion-advection model for two competing species in a heterogeneous environment. The species are assumed to be identical except their dispersal strategies: one disperses by random diffusion only, the other by both random diffusion and advection along environmental gradient. When the two competitors have the same diffusion rates and the strength of the advection is relatively weak in comparision to that of the random dispersal, we show that the competitor that moves toward more favorable environments has the competitive advantage, provided that the underlying spatial domain is convex, and the competitive advantage can be reversed for certain non-convex habitats. When the advection is strong relative to the dispersal, we show that both species can invade when they are rare, and the two competitors can coexist stably. The biological explanation is that for sufficiently strong advection, the "smarter" competitor will move toward more favorable environments and is concentrated at the place with maximum resources. This leaves enough room for the other species to survive since it can live upon regions with less resources.

1 Introduction

The semilinear parabolic system

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mu\Delta u + u[m(x) - u - v] && \text{in } \Omega \times (0,\infty), \\ \frac{\partial v}{\partial t} &= \nu\Delta v + v[m(x) - u - v] && \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0 && \text{on } \partial\Omega \times (0,\infty) \end{aligned} \tag{1.1}$$

models two competing species that are identical except for their migration rates. Here, the migration rates μ and ν are two positive constants, and $u(x,t)$ and $v(x,t)$ represent the densities of two species at location x and time t. The function $m(x)$ represents the intrinsic growth rates of species, and throughout this paper we assume that $m(x)$ is twice continuously differentiable in $\overline{\Omega}$. The habitat Ω is a bounded region in R^N, with smooth boundary $\partial\Omega$, n denotes the unit normal vector on $\partial\Omega$, and the no-flux boundary condition means that no individuals cross the boundary.

If we assume that the initial data $u(x,0)$ and $v(x,0)$ are non-negative and not identically zero, then by maximum principle [21], $u(x,t) > 0$ and $v(x,t) > 0$ for every $x \in \overline{\Omega}$ and every $t > 0$. Moreover, $u(x,t)$ and $v(x,t)$ are classical solutions of (1.1) and exist for all time $t > 0$. Of particular interest are the dynamics and coexistence states of (1.1). We say that a steady state (u_e, v_e) of (1.1) is a *coexistence state* if both components are positive, and it is a *semi-trivial state* if one component is positive and the other is zero.

We first make the following assumption on $m(x)$:

(A1) $m(x)$ is a non-constant function, and $\int_\Omega m > 0$.

Under assumption (A1), for every $\gamma > 0$, the scalar equation

$$\begin{aligned} \gamma\Delta\theta + (m-\theta)\theta &= 0 && \text{in } \Omega, \\ \frac{\partial\theta}{\partial n} &= 0 && \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

has a unique positive solution, denoted by $\theta(x,\gamma)$. This implies that (1.1) has two semi-trivial states, denoted by $(\theta(\cdot,\mu),0)$ and $(0,\theta(\cdot,\nu))$ for every $\mu > 0$ and every $\nu > 0$. It is shown in [9] that if $\mu < \nu$, then $(\theta(\cdot,\mu),0)$ is globally asymptotically stable among all non-negative non-trivial initial data. In other words, *the slower diffuser wins.* By symmetry, a similar conclusion holds when $\mu > \nu$. In particular, (1.1) has no coexistence states if $\mu \neq \nu$. For the case $\mu = \nu$, (1.1) has a family of coexistence states, which is the global attractor for all non-negative non-trivial initial data.

It seems reasonable to argue that besides the random dispersal, it is also very plausible that species could move upward along the resource gradient. See, e.g., [1, 2, 4, 6]

and references therein. In this paper we study the system

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nabla \cdot [\mu \nabla u - \alpha u \nabla m] + (m - u - v)u \qquad \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial t} &= \nabla \cdot [\nu \nabla v] + (m - u - v)v \qquad \text{in } \Omega \times (0, \infty), \end{aligned} \tag{1.3}$$

with no-flux boundary conditions

$$\mu \frac{\partial u}{\partial n} - \alpha u \frac{\partial m}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, \infty). \tag{1.4}$$

Here, the species with density v is assumed to disperse only by random diffusion, while the species with density u is to disperse by diffusion together with directed movement towards more favorable habitats (corresponding to $\alpha > 0$). Our primary goal is to understand the dynamics of (1.3)-(1.4) for large α. In particular, given arbitrary μ and ν and under very mild assumptions on $m(x)$, we will show that (1.3)-(1.4) has at least one stable coexistence state for large α. This is in strong contrast with the case $\alpha = 0$, for which there is no coexistence state and the slower diffuser is the sole winner.

When assumption (A1) holds, (1.3)-(1.4) has two semi-trivial states, denoted by $(\tilde{u}, 0)$ and $(0, \theta(\cdot, \nu))$, for every $\mu > 0$, every $\nu > 0$, and every $\alpha \geq 0$ (see [6]), where $\tilde{u}$ is the unique positive solution of

$$\begin{aligned} &\nabla \cdot [\mu \nabla \tilde{u} - \alpha \tilde{u} \nabla m] + (m - \tilde{u})\tilde{u} = 0 \qquad \text{in } \Omega, \\ &\mu \frac{\partial \tilde{u}}{\partial n} - \alpha \tilde{u} \frac{\partial m}{\partial n} = 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1.5}$$

For fixed μ, ν with $\mu \neq \nu$, the dynamics of (1.3)-(1.4) is similar to that of (1.1) for sufficiently small α. More precisely, there exists some small positive constant $\alpha_0 = \alpha_0(\mu, \nu, \Omega, m)$ such that if $\alpha \in (0, \alpha_0)$, $(\tilde{u}, 0)$ is the global attractor of (1.3)-(1.4) among all non-negative and non-trivial initial data if $\mu < \nu$, and $(0, \theta(\cdot, \nu))$ is the global attractor if $\mu > \nu$.

The case $\mu = \nu$ is quite delicate. This is due to the fact that (1.1) with $\mu = \nu$ is a degenerate system: it has a family of coexistence states, each of which is neutrally stable, and as a whole is a global attractor. As shown in recent studies [5, 17, 18], (1.1) with $\mu = \nu$ is very sensitive to perturbations, and the dynamics and coexistence states of (1.1) after perturbations can be very complex. For sufficiently small positive α, (1.3)-(1.4) can also be viewed as a perturbation of (1.1).

For $\mu > 0$, define

$$\alpha^*(\mu) = \frac{\displaystyle\int_\Omega \theta(x, \mu) \nabla \theta(x, \mu) \cdot \nabla m(x)\, dx}{\displaystyle\int_\Omega |\nabla \theta(x, \mu)|^2\, dx}.$$

As will be seen later, this quantity plays a crucial role in studying dynamics of (1.3)-(1.4) for small positive α.

For any $\mu_0 > 0$, $\mu_1, \nu_1 \in R^1$, and $\alpha_1 > 0$, let

$$(\mu, \nu, \alpha) = (\mu_0 + \mu_1 s + o(s), \mu_0 + \nu_1 s + o(s), \alpha_1 s + o(s)), \tag{1.6}$$

where s is positive and small.

Theorem 1.1 *Suppose that* (A1) *holds and* Ω *is convex. Then*

a) *for every* $\mu > 0$, $\alpha^*(\mu) > 0$.

b) *Let* μ, ν, α *be given as in* (1.6). *If* $\alpha_1 > (\mu_1 - \nu_1)/\alpha^*(\mu_0)$, *then for positive small* s, $(\tilde{u}, 0)$ *is globally asymptotically stable. In particular, if* $(\mu, \nu) = (\mu_0, \mu_0)$, $(\tilde{u}, 0)$ *is globally asymptotically stable for small positive* α.

Theorem 1.1 is established in [5] except the global convergence conclusion in part (b), i.e., every solution of (1.3)-(1.4) satisfies $(u, v) \to (\tilde{u}, 0)$ as $t \to \infty$. Theorem 1.1 has some interesting consequences: e.g., for the case $\mu_1 > \nu_1$, it implies that the competitor that moves toward more favorable environments may have a competitive advantage even if it diffuses more rapidly than the other competitor. This is in strong contrast with the case in which both competitors disperse only by random diffusion, where the slow diffuser always wins. It means that the advantage gained from the directed movement upward resource gradients can counterbalance the disadvantage created by faster diffusion.

We should point out that the convexity assumption on domain Ω in Theorem 1.1 seems to be necessary, as shown by the following result.

Theorem 1.2 *Given any* $\mu_0 > 0$, *there exist non-convex domain* Ω *and smooth function* $m(x)$ *such that:*

a) $\alpha^*(\mu_0) < 0$, *and* $\alpha^*(\mu)$ *changes sign at least once in* $(0, \mu_0)$.

b) *Let* μ, ν, α *be given by* (1.6). *If* $\alpha_1 > (\mu_1 - \nu_1)/\alpha^*(\mu_0)$, *then for positive small* s, $(0, \theta(\cdot, \nu))$ *is globally asymptotically stable. In particular, if* $(\mu, \nu) = (\mu_0, \mu_0)$, $(0, \theta(\cdot, \mu_0))$ *is globally asymptotically stable for small positive* α.

For the case $\mu_1 < \nu_1$, part (b) of Theorem 1.2 implies that for certain non-convex habitats, a slower diffuser which also moves toward more favorable environments may not have the competitive advantage. This is in strong contrast with both the case of convex habitat and the case of $\alpha = 0$.

The main goal of this paper is to study the much more interesting and challenging case when α is large, and show how strong advection can induce stable coexistence of competing species. In particular, we shall investigate the stability of $(\tilde{u}, 0)$ and $(0, \theta(\cdot, \nu))$, and the existence and qualitative properties of coexistence states.

The stability of $(\tilde{u}, 0)$ and properties of coexistence states rely crucially on qualitative properties of $\tilde{u}$. To this end we first make the following assumption:

(A2) The set of critical points of $m(x)$ has Lebesgue measure zero.

Theorem 1.3 *Suppose that* (A1) *is satisfied.*

a) *If* (A2) *holds, then* $\|\tilde{u}\|_{L^2(\Omega)} \to 0$ *as* $\alpha \to \infty$.

b) *If* $m(x) > 0$ *in* $\overline{\Omega}$ *and* $\alpha > \mu / \min_{\overline{\Omega}} m$, *then*

$$\tilde{u}(x) \geq \max_{\overline{\Omega}} m \cdot e^{(\alpha/\mu)[m(x) - \max_{\overline{\Omega}} m]} \tag{1.7}$$

for every $x \in \overline{\Omega}$. *In particular,* $\max_{\overline{\Omega}} \tilde{u} \geq \max_{\overline{\Omega}} m$.

Theorem 1.3 implies that if $m(x) > 0$ in $\overline{\Omega}$ and (A2) holds, then for sufficiently large α, $\tilde{u}$ is concentrated at the global maxima of function $m(x)$. It is natural to inquire whether $\tilde{u}$ can concentrate at other locations. In this connection, we make the following assumption.

(A3) Suppose that $\Omega = (0,1)$, $m_x(0) \geq 0 \geq m_x(1)$, and $m(x)$ has finitely many critical points in $[0,1]$, denoted by $\{x_1, ..., x_k\}$.

Theorem 1.4 *Suppose that* (A1) *holds and* $\Omega = (0,1)$.

a) *If* $m'(x) > 0$ *in* $[0,1]$, *then for sufficiently large* α, $\tilde{u}'(x) > 0$ *in* $[0,1]$, $\tilde{u}(x) \to 0$ *uniformly in* $[0,c]$ *for every* $c \in (0,1)$, *and* $u(1) \geq \int_0^1 m > 0$.

b) *If* (A3) *holds, then* $\tilde{u}(x) \to 0$ *uniformly in every compact subset of* $[0,1] \setminus \{x_1, ..., x_k\}$ *as* $\alpha \to \infty$. *In particular,* $\tilde{u}(x) \to 0$ *pointwise for every* $x \in [0,1] \setminus \{x_1, ..., x_k\}$ *as* $\alpha \to \infty$.

We conjecture that $\tilde{u}$ is concentrated only at local maxima of function $m(x)$. This conjecture is confirmed by part (a) for the special case when $m(x)$ has no critical points, and is also partially supported by part (b) of Theorem 1.4, which says that $\tilde{u}$ concentrates only at critical points of $m(x)$.

For the stability of $(0, \theta(\cdot, \nu))$, we assume that $m(x)$ has at least one isolated global maximum as follows:

(A4) There exists some $x_0 \in \overline{\Omega}$ and $\delta > 0$ such that $m(x_0) = \max_{\overline{\Omega}} m$ and $m(x_0) > m(x)$ for every $x \in \overline{B_\delta(x_0) \cap \Omega} \setminus \{x_0\}$.

For sufficiently large α, we have the following result.

Theorem 1.5 *Suppose that assumption* (A1) *is satisfied.*

a) *If* (A2) *holds, then for every* $\mu > 0$, *there exists some positive constant* $\alpha_2 = \alpha_2(\mu, m, \Omega)$ *such that if* $\alpha \geq \alpha_2$, $(\tilde{u}, 0)$ *is unstable for every* $\nu > 0$.

b) *If* (A4) *holds, then for every* $\mu > 0$ *and* $\eta > 0$, *there exists some positive constant* $\alpha_3 = \alpha_3(\mu, \eta, m, \Omega)$ *such that if* $\alpha \geq \alpha_3$, $(0, \theta(\cdot, \nu))$ *is unstable for every* $\nu \geq \eta$.

c) *If* (A2) *and* (A4) *hold, then for every* $\mu > 0$ *and* $\eta > 0$, *there exists some positive constant* $\alpha_4 = \alpha_4(\mu, \eta, m, \Omega)$ *such that if* $\nu \geq \eta$ *and* $\alpha \geq \alpha_4$, *system* (1.3)-(1.4) *has at least one stable coexistence state.*

d) *If* (A2) *holds, every coexistence state* (u_α, v_α) *of* (1.3)-(1.4) *satisfies* $u_\alpha \to 0$ *in* $L^2(\Omega)$ *and* $v_\alpha \to \theta(\cdot, \nu)$ *in* $W^{2,2}(\Omega)$*; if* (A3) *holds, then* $u_\alpha \to 0$ *pointwise for every* $x \in [0,1] \setminus \{x_1, ..., x_k\}$.

Remark 1.6 *Since* (1.3)-(1.4) *is a strongly monotone system (Lemma* 2.2*), as in other competition models, the existence and stability of coexistence states in* (c) *follow from the instability results on the two semi-trivial states in* (a) *and* (b) *and theory for continuous monotone systems* [7, 13, 20]. *If we further assume that* $m(x)$ *is analytic, then* (1.3)-(1.4) *has at least one asymptotically stable coexistence state* [14]. *For discrete-time counterpart of results for monotone systems, we refer to* [8, 12] *and references therein.*

Parts (a) and (c) of Theorem 1.5 are somewhat surprising. For fixed $\mu < \nu$, the species u always wins provided that $\alpha > 0$ is sufficiently small. As α increases, since the species u has the tendency to move toward more favorable regions, it should have more competitive advantage and should still be the sole winner of the competition. However, part (a) implies that species with density v can invade when rare, and part (c) illustrates that the two species can coexist stably for large α. This is in strong contrast with the case when $\alpha = 0$ or $\alpha > 0$ is sufficiently small.

Part (d) seems to offer a possible explanation for the existence of stable coexistence states for large α. Namely, as α becomes sufficiently large, the species u tends to concentrate around critical points of $m(x)$, and this leaves sufficient resources for the other species to survive. We conjecture that for sufficiently large α, (1.3)-(1.4) has a unique coexistence state, denoted by (u_α, v_α), which is globally asymptotically stable among non-negative non-trivial initial data. Moreover, as $\alpha \to \infty$, u_α concentrates at global maxima of $m(x)$ in $\overline{\Omega}$.

This paper is organized as follows. In Section 2 we consider the case when α is positive small, and Theorems 1.1 and 1.2 will be established therein. Section 3 is devoted to the study of qualitative properties of $\tilde{u}$ for arbitrary or large α, and Theorems 1.3 and 1.4 will be proved. In Section 4 we investigate the stability of two semi-trivial states and establish Theorems 1.5.

2 The case $0 < \alpha \ll 1$

In this section we consider the dynamics of (1.3)-(1.4) when α is positive and sufficiently small. Theorem 1.1 will be established in Subsect. 2.1, and Subsect. 2.2 is devoted to the proof of Theorem 1.2.

2.1 Convex domains

In this subsection we study (1.3)-(1.4) for sufficiently small α when the underlying domain Ω is convex.

Lemma 2.1 *Suppose that* m *is non-constant. Let* (μ, ν, α) *be given by* (1.6). *If* $\alpha^*(\mu_0) \neq 0$, $\alpha_1 \neq (\mu_1 - \nu_1)/\alpha^*(\mu_0)$ *and* (1.2) *with* $\gamma = \mu_0$ *has a positive solution, then* (1.3)-(1.4) *has no coexistence state for positive small* s.

Note that in Lemma 2.1 we do not assume that Ω is convex. This generality will be needed in Subsect. 2.2.

Proof. We argue by contradiction. Suppose that (1.3)-(1.4) has a coexistence state (u_s, v_s) for every sufficiently small positive s. By elliptic regularity [10], passing to subsequence if necessary, we may assume that $(u_s, v_s) \to (u^*, v^*)$ as $s \to 0$, where $u^* \geq 0$ and $v^* \geq 0$ in $\overline{\Omega}$, and (u^*, v^*) satisfies

$$\begin{aligned} &\mu_0 \Delta u^* + u^*(m - u^* - v^*) = 0 && \text{in } \Omega, \\ &\mu_0 \Delta v^* + v^*(m - u^* - v^*) = 0 && \text{in } \Omega, \\ &\frac{\partial u^*}{\partial n} = \frac{\partial v^*}{\partial n} = 0 && \text{on } \Omega. \end{aligned} \tag{2.1}$$

Hence, $u^* + v^*$ satisfies

$$\begin{aligned} &\mu_0 \Delta (u^* + v^*) + (u^* + v^*)(m - u^* - v^*) = 0 && \text{in } \Omega, \\ &\frac{\partial (u^* + v^*)}{\partial n} = 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

Therefore, either $u^* + v^* \equiv 0$ or $u^* + v^* \equiv \theta(\cdot, \mu_0)$. We show that the only possibility is $u^* + v^* \equiv \theta(\cdot, \mu_0)$. If $u^* + v^* \equiv 0$, i.e., $u^* \equiv v^* \equiv 0$, we have $(u_s, v_s) \to (0, 0)$ uniformly in x as $s \to 0$. Set $\hat{v}_s = v_s / \|v_s\|_{L^\infty(\Omega)}$. By elliptic regularity we may assume that $\hat{v}_s \to \hat{v}$ in $C^2(\overline{\Omega})$, where $\hat{v}$ is non-trivial, non-negative and satisfies

$$\mu_0 \Delta \hat{v} + m\hat{v} = 0 \quad \text{in } \Omega, \quad \frac{\partial \hat{v}}{\partial n}|_{\partial\Omega} = 0. \tag{2.3}$$

Multiplying (2.3) by $\theta(\cdot, \mu_0)$, integrating in Ω and applying (1.2) with $\gamma = \mu_0$, we have

$$\int_\Omega \theta^2(x, \mu_0) \hat{v}(x)\, dx = 0,$$

which is a contradiction since both $\theta(\cdot, \mu_0)$ and $\hat{v}$ are positive. Hence, $u^* + v^* \equiv \theta(\cdot, \mu_0)$.

There are three possibilities for us to consider:

Case 1. $u^* \equiv 0$ and $v^* = \theta(\cdot, \mu_0)$. For this case, we define $\hat{u}_s = u_s / \|u_s\|_{L^\infty(\Omega)}$. Then $\hat{u}_s$ satisfies

$$\nabla \cdot [\mu \nabla \hat{u}_s - \alpha \hat{u}_s \nabla m] + \hat{u}_s(m - u_s - v_s) = 0 \quad \text{in } \Omega$$

and the no-flux boundary condition. Hence, by elliptic regularity, we may assume that $\hat{u}_s \to \hat{u}^*$ in $C^2(\overline{\Omega})$, and $\hat{u}^*$ satisfies $\max_{\overline{\Omega}} \hat{u}^* = 1$, $\hat{u}^* \geq 0$, and

$$\mu_0 \Delta \hat{u}^* + \hat{u}^*[m - \theta(\cdot, \mu_0)] = 0 \quad \text{in } \Omega, \quad \partial \hat{u}^* / \partial n|_{\partial\Omega} = 0.$$

Therefore, $\hat{u}^* \equiv \theta(\cdot, \mu_0) / \|\theta(\cdot, \mu_0)\|_{L^\infty}$.

Multiplying the equation of u_s by v_s and the equation of v_s by u_s, subtracting and integrating in Ω, we have

$$\alpha \int_\Omega u_s \nabla m \cdot \nabla v_s = (\mu - \nu) \int_\Omega \nabla u_s \cdot \nabla v_s. \tag{2.4}$$

Applying (1.6), dividing both sides of (2.4) by s and $\|u_s\|_{L^\infty(\Omega)}$, we obtain

$$(\alpha_1 + o(1)) \int_\Omega \hat{u}_s \nabla m \cdot \nabla v_s = (\mu_1 - \nu_1 + o(1)) \int_\Omega \nabla \hat{u}_s \cdot \nabla v_s. \tag{2.5}$$

Letting $s \to 0$ in (2.5) we have

$$\alpha_1 \int_\Omega \theta(\cdot, \mu_0) \nabla m \cdot \nabla \theta(\cdot, \mu_0) = (\mu_1 - \nu_1) \int_\Omega |\nabla \theta(\cdot, \mu_0)|^2, \tag{2.6}$$

i.e., $\alpha_1 = (\mu_1 - \nu_1)/\alpha^*(\mu_0)$, which contradicts our assumption.

Case 2. $v^* \equiv 0$ and $u^* \equiv \theta$. Since the proof of this case is similar to that of case 1, it is omitted.

Case 3. $u^* > 0$ and $v^* > 0$. By $u^* + v^* \equiv \theta(\cdot, \mu_0)$ and (2.1) we see that $(u^*, v^*) = (\tau\theta(\cdot, \mu_0), (1 - \tau\theta(\cdot, \mu_0))$ for some $\tau \in (0, 1)$. Dividing (2.4) by s and passing to the limit, we see that (2.6) again holds. This contradiction completes the proof. □

Lemma 2.2 *Let $(u_i(x,t), v_i(x,t))$, $i = 1, 2$, be two solutions of* (1.3)-(1.4), *$u_1(x,0) \geq u_2(x,0)$ and $v_1(x,0) \leq v_2(x,0)$ for every $x \in \Omega$. Then $u_1(x,t) \geq u_2(x,t)$ and $v_1(x,t) \leq v_2(x,t)$ for every $x \in \Omega$ and $t > 0$. If further assume that $u_1(x,0) \not\equiv u_2(x,0)$ and $v_1(x,0) \not\equiv v_2(x,0)$, then $u_1(x,t) > u_2(x,t)$ and $v_1(x,t) < v_2(x,t)$ for every $x \in \overline{\Omega}$ and every $t > 0$.*

Proof. Set $w = e^{(-\alpha/\mu)m} u$. Then (1.3)-(1.4) become

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \mu \Delta w + \alpha \nabla m \cdot \nabla w + \left[m - e^{(\alpha/\mu)m} w - v \right] w && \text{in } \Omega \times (0, \infty), \\
\frac{\partial v}{\partial t} &= \nu \Delta v + \left[m - e^{(\alpha/\mu)m} w - v \right] v && \text{in } \Omega \times (0, \infty), \\
\frac{\partial w}{\partial n} &= \frac{\partial v}{\partial n} = 0 && \text{on } \partial\Omega \times (0, \infty).
\end{aligned} \tag{2.7}$$

Since $w_1(x,0) \geq w_2(x,0)$, $v_1(x,0) \leq v_2(x,0)$, and (2.7) is a monotone system [3, 12, 15], we have $w_1(x,t) \geq w_2(x,t)$ and $v_1(x,t) \leq v_2(x,t)$. The rest of the proof follows similarly from the maximum principle. This completes the proof. □

We are now ready for

Proof of Theorem 1.1. By Theorem 3.3 of [4], if $\alpha_1 > (\mu_1 - \nu_1)/\alpha^*(\mu_0)$, $(\tilde{u}, 0)$ is stable and $(0, \theta(\cdot, \nu))$ is unstable for positive small s. By Lemmas 2.1 and 2.2 and theory for monotone systems [12, 15], we see that $(\tilde{u}, 0)$ is globally asymptotically stable for positive small s. □

2.2 Non-convex domains

In this subsection we consider the dynamics of (1.3)-(1.4) for certain non-convex domains Ω and small positive α.

Lemma 2.3 *Fix any $\hat{\mu} > 0$. There exist a non-convex domain $\Omega \subset R^2$ and a smooth function $m(x)$ such that (1.2) has a positive solution for $0 < \gamma \leq \hat{\mu}$, and $\alpha^*(\hat{\mu}) < 0$.*

For $0 < \epsilon \ll 1$, define

$$\Omega_\epsilon = \{(x_1, x_2) : 0 < x_1 < 1, 0 < x_2 < \epsilon a(x_1)\},$$

where $a(x_1)$ is some positive smooth function in [0, 1] to be chosen. We assume $m(x) = m(x_1)$, which will also be chosen later.

We first choose some smooth function $\theta_1(z) : 0 \leq z \leq 1$ such that it satisfies (i) $\theta_1(z) > 0$ in $[0, 1]$; (ii) $\theta_{1,z}(0) = \theta_{1,z}(1) = 0$; (iii) $\theta_{1,zz} \cdot (\theta_1^2)_{zz}$ is negative somewhere in $(0, 1)$. For constructions of such a function θ_1, see ψ_1 in Lemma 3.2 of [6].

Next, we can choose a smooth function $a_1(z)$ such that it satisfies (i) $a_1(z) > 0$ in $[0, 1]$, (ii) $\int_0^1 a_1(z)\, dx = 1$, and (iii)

$$\hat{\mu} \int_0^1 \frac{1}{a_1^2} \frac{\theta_{1,zz} \cdot (\theta_1^2)_{zz}}{\theta_1}\, dz < -2 \int_0^1 \theta_1 \theta_{1,z}^2\, dz. \tag{2.8}$$

Set $y_1 = \int_0^z a_1(s)\, ds$. Since $dy_1/dz = a_1(z) > 0$, we can write $z = z(y_1)$. Define

$$a(y_1) = a_1(z(y_1)), \quad \tilde{\theta}(y_1) = \theta_1(z(y_1)), \quad m(y_1) = m_1(z(y_1)), \tag{2.9}$$

where

$$m_1(z) = \theta_1 - \frac{\hat{\mu}\theta_{1,zz}}{\theta_1 a_1^2}.$$

Hence, θ_1 satisfies

$$\begin{aligned} &\hat{\mu}\theta_{1,zz} + a_1^2\theta_1[m_1(z) - \theta_1] = 0 \quad \text{in } (0, 1), \\ &\theta_{1,z}(0) = \theta_{1,z}(1) = 0. \end{aligned} \tag{2.10}$$

By the change of variable $z = z(y_1)$, we see that $\tilde{\theta}$ is the unique positive solution of the equation:

$$\begin{aligned} &\hat{\mu}\frac{d}{dy_1}\left(a\frac{d\tilde{\theta}}{dy_1}\right) + a\tilde{\theta}(m - \tilde{\theta}) = 0 \quad \text{in } (0, 1), \\ &\tilde{\theta}_{y_1}(0) = \tilde{\theta}_{y_1}(1) = 0. \end{aligned} \tag{2.11}$$

Claim. For such choices of m and a, (1.2) has a unique positive solution for all $\gamma \in (0, \hat{\mu}]$. If $\int_{\Omega_\epsilon} m \geq 0$, (1.2) has a unique positive solution for all $\gamma > 0$, so there is nothing to prove. If $\int_{\Omega_\epsilon} m < 0$, let $\mu_\epsilon > 0$ denote the principal eigenvalue of the linearized equation of (1.2), i.e., the equation

$$\mu_\epsilon \Delta\varphi + m\varphi = 0 \quad \text{in } \Omega_\epsilon, \quad \frac{\partial\varphi}{\partial n_\epsilon}|_{\partial\Omega_\epsilon} = 0 \tag{2.12}$$

has a positive solution, where n_ϵ is the outward unit normal vector on $\partial\Omega_\epsilon$. It is well known that (1.2) has a positive solution if and only if $\gamma \in (0, \mu_\epsilon)$. By Lemma 3.6 and

the proof of Theorem 3.1 of [6], we see that $\mu_\epsilon \to \mu^*$ as $\epsilon \to 0$, where $\mu^* > 0$ is the principal eigenvalue of the equation

$$\begin{aligned} &\mu \frac{d}{dy_1}\left(a\frac{d\varphi}{dy_1}\right) + am\varphi = 0 \quad \text{in } (0,1), \\ &\varphi_{y_1}(0) = \varphi_{y_1}(1) = 0. \end{aligned} \tag{2.13}$$

Since (2.11) has a positive solution, we see that $\mu^* > \hat{\mu}$. This implies that for small positive ϵ, $\mu_\epsilon > \hat{\mu}$. Hence, (1.2) has a positive solution for all $\gamma \in (0, \hat{\mu}]$.

Let θ^ϵ denote the unique positive solution of (1.2) with $\gamma = \hat{\mu}$, $m = m(x_1)$, and $\Omega = \Omega_\epsilon$, i.e.,

$$\begin{aligned} &\hat{\mu}\Delta\theta^\epsilon + \theta^\epsilon\left[m(x_1) - \theta^\epsilon\right] = 0 && \text{in } \Omega_\epsilon, \\ &\frac{\partial\theta^\epsilon}{\partial n_\epsilon} = 0 && \text{on } \partial\Omega_\epsilon. \end{aligned} \tag{2.14}$$

We introduce the transformation

$$x_1 = y_1, \quad x_2 = \epsilon a(y_1) y_2.$$

Under this new coordinate, Ω_ϵ becomes $\Omega = (0,1) \times (0,1)$. Set $\theta_\epsilon(y_1, y_2) = \theta^\epsilon(x_1, x_2)$. Then θ_ϵ satisfies

$$\begin{aligned} &\hat{\mu}\nabla \cdot (B_\epsilon \theta_\epsilon) + a\theta_\epsilon\left[m(y_1) - \theta_\epsilon\right] = 0 && \text{in } \Omega, \\ &B_\epsilon \theta_\epsilon \cdot n = 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.15}$$

where B_ϵ is given by

$$B_\epsilon u = \left(a u_{y_1} - a_{y_1} y_2 u_{y_2},\ -a_{y_1} y_2 u_{y_1} + \frac{1 + \epsilon^2 a_{y_1}^2 y_2^2}{a\epsilon^2} u_{y_2}\right),$$

and n is the unit normal vector on $\partial\Omega$. Multiplying (2.15) by θ_ϵ and integrating in Ω, as in [11] we have

$$\int_\Omega \left[a\left(\theta_{\epsilon,y_1} - \frac{a_{y_1}}{a} y_2 \theta_{\epsilon,y_2}\right)^2 + \frac{\theta_{\epsilon,y_2}^2}{\epsilon^2 a}\right] \le C$$

for some positive constant C which is independent of ϵ. Hence, $\|\theta_\epsilon\|_{W^{1,2}(\Omega)} \le C$ and $\|\theta_{\epsilon,y_2}\|_{L^2(\Omega)} \le C\epsilon$. Therefore, $\theta_\epsilon \to \hat{\theta}$ weakly in $W^{1,2}(\Omega)$, and $\hat{\theta} \ge 0$ a.e. in Ω. Since $\|\theta_{\epsilon,y_2}\|_{L^2(\Omega)} \to 0$ as $\epsilon \to 0$, we have $\hat{\theta}_{y_2} = 0$ a.e., which implies that $\hat{\theta}(y) = \hat{\theta}(y_1)$ a.e.

Multiplying (2.15) by any $\eta = \eta(y_1) \in W^{1,2}(0,1)$, using integration by parts we have

$$-\hat{\mu}\int_\Omega \eta_{y_1}(a\theta_{\epsilon,y_1} - a_{y_1} y_2 \theta_{\epsilon,y_2}) dy_1 dy_2 + \int_\Omega a\eta\theta_\epsilon[m(y_1) - \theta_\epsilon] dy_1 dy_2 = 0.$$

Let $\epsilon \to 0$ we have

$$-\hat{\mu}\int_0^1 \eta_{y_1} a(y_1)\hat{\theta}_{y_1} dy_1 + \int_0^1 a\eta\hat{\theta}(m - \hat{\theta}) dy_1 = 0,$$

which implies that $\hat{\theta}$ is a smooth solution of (2.11).

We further show that $\hat{\theta} \not\equiv 0$. We argue by contradiction. If not, we have $\theta_\epsilon \to 0$ weakly in $W^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$. Set $\theta_\epsilon^* = \theta_\epsilon / \|\theta_\epsilon\|_{L^2(\Omega)}$. Then $\|\theta_\epsilon^*\|_{L^2(\Omega)} = 1$. By a similar argument as before, we have $\theta_\epsilon^* \to \theta^*$ weakly in $W^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$, where $\theta^* \geq 0$ a.e. in Ω and is a smooth solution of

$$\begin{aligned} &\hat{\mu}\frac{d}{dy_1}\left(a\frac{d\theta^*}{dy_1}\right) + am\theta^* = 0 \quad \text{in } (0,1), \\ &\theta^*_{y_1}(0) = \theta^*_{y_1}(1) = 0, \quad \|\theta^*\|_{L^2(\Omega)} = 1. \end{aligned} \tag{2.16}$$

However, this is impossible since (2.11) has a positive solution $\tilde{\theta}$. This contradiction implies that $\hat{\theta} \not\equiv 0$. Since (2.11) has a unique positive solution, we have $\hat{\theta} \equiv \tilde{\theta}$.

Proof of Lemma 2.3. By previous discussions, it remains to show that $\alpha^*(\hat{\mu}) < 0$. Since

$$\theta^\epsilon_{x_1} = \theta_{\epsilon,y_1} - \frac{a_{y_1}}{a} y_2 \theta_{\epsilon,y_2},$$

we have

$$\begin{aligned} &\int_{\Omega_\epsilon} \theta^\epsilon \nabla m \cdot \nabla \theta^\epsilon \, dx_2 dx_1 \\ &= \int_0^1 \int_0^{\epsilon a(x_1)} \theta^\epsilon m_{x_1} \theta^\epsilon_{x_1} \, dx_1 dx_2 \\ &= \epsilon \int_0^1 \int_0^1 a(y_1) m_{y_1} \theta_\epsilon \left[\theta_{\epsilon,y_1} - \frac{a_{y_1}}{a} y_2 \theta_{\epsilon,y_2}\right] dy_1 dy_2. \end{aligned} \tag{2.17}$$

Since $\|\theta_{\epsilon,y_2}\|_{L^2(\Omega)} \to 0$ and $\theta_\epsilon \to \tilde{\theta}$ weakly in $W^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$, we have

$$\begin{aligned} \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{\Omega_\epsilon} \theta^\epsilon \nabla m \cdot \nabla \theta^\epsilon \, dx_1 dx_2 &= \int_0^1 a m_{y_1} \tilde{\theta} \tilde{\theta}_{y_1} \, dy_1 \\ &= \int_0^1 m_{1,z} \theta_1 \theta_{1,z} \, dz \\ &= -\frac{1}{2} \int_0^1 m_1 (\theta_1^2)_{zz} \, dz \\ &= \frac{\hat{\mu}}{2} \int_0^1 \frac{1}{a_1^2} \frac{\theta_{1,zz} (\theta_1^2)_{zz}}{\theta_1} + \int_0^1 \theta_1 (\theta_{1,z})^2 \, dz \\ &< 0, \end{aligned} \tag{2.18}$$

where the last inequality follows from (2.8). Therefore, for any $\hat{\mu} > 0$, by choosing functions a and m suitably, we have $\alpha^*(\hat{\mu}) < 0$ for sufficiently positive small ϵ. This completes the proof of Lemma 2.3. $\square$

Lemma 2.4 *Suppose that m is non-constant and positive somewhere in Ω. Then*

$$\lim_{\mu\to 0} \int_\Omega \theta(x,\mu) \nabla \theta(x,\mu) \cdot \nabla m \, dx = \int_{\{m>0\}} m |\nabla m|^2 > 0.$$

Proof. By integration by parts we have

$$\begin{aligned}\int_\Omega \theta\nabla\theta\cdot\nabla m &= \frac{1}{2}\int_\Omega \nabla(\theta)^2\cdot\nabla m \\ &= \frac{1}{2}\int_{\partial\Omega}\theta^2\frac{\partial m}{\partial n} - \frac{1}{2}\int_\Omega \theta^2\Delta m.\end{aligned} \tag{2.19}$$

It is known that $\theta(x,\mu)\to m_+(x) = \max\{m(x),0\}$ uniformly as $\mu\to 0$ [16]. Since $m_+\in W^{1,2}(\Omega)$, $\nabla m_+ = \nabla m$ for $m(x)>0$, and $\nabla m_+ = 0$ for $m(x)\le 0$, we have

$$\begin{aligned}\lim_{\mu\to 0}\int_\Omega \theta\nabla\theta\cdot\nabla m &= \frac{1}{2}\int_{\partial\Omega}(m_+)^2\frac{\partial m}{\partial n} - \frac{1}{2}\int_\Omega (m_+)^2\Delta m \\ &= \frac{1}{2}\int_\Omega \nabla(m_+)^2\cdot\nabla m \\ &= \int_{\{m>0\}} m|\nabla m|^2.\end{aligned} \tag{2.20}$$

□

Proof of Theorem 1.2. Part (a) follows from Lemmas 2.3 and 2.4. Since $\alpha^*(\hat\mu)<0$, by (3.18) of [5] we know that if $\alpha_1 > (\mu_1-\nu_1)/\alpha^*(\hat\mu)$, $(\tilde u, 0)$ is unstable and $(0,\theta(\cdot,\nu))$ is stable for small positive s. Since $\alpha^*(\hat\mu)\neq 0$, by Lemma 2.1 we see that (1.3)-(1.4) with (μ,ν,α) given by (1.6) has no coexistence states for small positive s. By Lemma 2.2 and theory for monotone systems [12, 15], $(0,\theta(\cdot,\nu))$ is globally asymptotically stable. □

3 Qualitative properties of $\tilde u$

In this section we study qualitative properties of $\tilde u$ for either arbitrary α or sufficiently large α. Such properties play essential roles in later studies of stability of $(\tilde u, 0)$ and asymptotic behaviors of coexistence states. It is easy to see that Theorem 1.3 follows from Lemma 3.2 and Theorem 3.5.

3.1 Preliminary bounds of $\tilde u$

We first establish some uniform bounds for $\tilde u$ for arbitrary or large α.

Lemma 3.1 *The following estimate holds:*

$$\|\tilde u\|_{L^2(\Omega)} \le \|m\|_{L^2(\Omega)}. \tag{3.1}$$

Proof. Integrating (1.5) in Ω, by Cauchy-Schwartz inequality we have

$$\int_\Omega \tilde u^2 = \int_\Omega m\tilde u \le \|m\|_{L^2(\Omega)}\|\tilde u\|_{L^2(\Omega)},$$

from which (3.1) follows. □

Lemma 3.2 *Suppose that $m(x) > 0$ in $\overline{\Omega}$. If $\alpha > \mu/\min_{\overline{\Omega}} m$, inequality (1.7) holds for every $x \in \overline{\Omega}$.*

Proof. Set $w = \tilde{u} \cdot e^{-(\alpha/\mu)m}$. Then w satisfies

$$\begin{aligned} &\mu\Delta w + \alpha\nabla m \cdot \nabla w + w\left[m - e^{(\alpha/\mu)m} w\right] = 0 \quad \text{in } \Omega, \\ &\frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.2}$$

Assume that $\min_{\overline{\Omega}} w = w(x_\alpha)$ for some $x_\alpha \in \overline{\Omega}$. By Proposition 3.2 of [19] we have

$$w(x_\alpha) \geq m(x_\alpha) e^{-(\alpha/\mu)m(x_\alpha)}.$$

Set $h(y) = ye^{-(\alpha/\mu)y}$. It is easy to check that $h' < 0$ for $y \in (\mu/\alpha, +\infty)$. Since $m(x_\alpha) \in [\min_{\overline{\Omega}} m, \max_{\overline{\Omega}} m]$, we have

$$w(x_\alpha) \geq \max_{\overline{\Omega}} m \cdot e^{-(\alpha/\mu)\max_{\overline{\Omega}} m}.$$

By the choice of x_α, we have

$$w(x) \geq \max_{\overline{\Omega}} m \cdot e^{-(\alpha/\mu)\max_{\overline{\Omega}} m},$$

from which (1.7) follows. □

Lemma 3.3 *Suppose that $\partial m/\partial n \leq 0$ on $\partial\Omega$. Then*

$$\|\tilde{u}\|_{L^\infty(\Omega)} \leq \|m\|_{L^\infty(\Omega)} + \alpha\|\Delta m\|_{L^\infty(\Omega)}. \tag{3.3}$$

Proof. Rewrite (1.5) as

$$\mu\Delta\tilde{u} - \alpha\nabla\tilde{u} \cdot \nabla m + f(x, \tilde{u}) = 0$$

in Ω, where $f(x, u) = \tilde{u}[m - \alpha\Delta m - \tilde{u}]$. Suppose that $\tilde{x}$ satisfies $\tilde{u}(\tilde{x}) = \max_{\overline{\Omega}} \tilde{u}$. Since $\partial\tilde{u}/\partial n \leq 0$ on $\partial\Omega$, by Proposition 3.2 of [19], we have $f(\tilde{x}, \tilde{u}(\tilde{x})) \geq 0$, from which (3.3) follows immediately. □

Lemma 3.4 *Suppose that $\partial m/\partial n \leq 0$ on $\partial\Omega$. Then there exists some constant C, independent of α, such that*

$$\int_\Omega \tilde{u}|\nabla m|^2 \leq \frac{C}{\alpha}. \tag{3.4}$$

Proof. Multiplying (1.5) by m, integrating in Ω, we have

$$-\int_\Omega \nabla m \cdot [\mu\nabla\tilde{u} - \alpha\tilde{u}\nabla m] + \int_\Omega m\tilde{u}(m - \tilde{u}) = 0.$$

Since

$$\int_\Omega \nabla\tilde{u} \cdot \nabla m = -\int_\Omega \tilde{u}\Delta m + \int_{\partial\Omega} \tilde{u}\frac{\partial m}{\partial n} \leq -\int_\Omega \tilde{u}\Delta m,$$

we have

$$\int_\Omega \tilde{u}|\nabla m|^2 \leq \frac{1}{\alpha}\int_\Omega \left[\tilde{u}(-\mu\Delta m - m^2) + m\tilde{u}^2\right]. \tag{3.5}$$

(3.4) follows from Lemma 3.1 and (3.5). □

3.2 L^2 convergence of $\tilde{u}$

In this subsection we establish part (a) of Theorem 1.3.

Theorem 3.5 *If* (A1) *and* (A2) *hold, then* $\int_\Omega \tilde{u}^2 \to 0$ *as* $\alpha \to \infty$.

Proof. Multiplying (1.5) by $\varphi \in \mathcal{S}$, where

$$\mathcal{S} \equiv \left\{ \varphi \in C^2(\overline{\Omega}) : \ \frac{\partial \varphi}{\partial n}|_{\partial\Omega} = 0 \right\},$$

and integrating in Ω, we have

$$-\mu \int_\Omega \nabla\varphi \cdot \nabla\tilde{u} + \alpha \int_\Omega \tilde{u}\nabla m \cdot \nabla\varphi = \int_\Omega \varphi\tilde{u}(\tilde{u} - m).$$

By the boundary condition of φ,

$$\int_\Omega \nabla\varphi \cdot \nabla\tilde{u} = -\int_\Omega \tilde{u}\Delta\varphi.$$

Hence

$$\mu \int_\Omega \tilde{u}\Delta\varphi + \alpha \int_\Omega \tilde{u}(\nabla\varphi \cdot \nabla m) = \int_\Omega \varphi\tilde{u}(\tilde{u} - m). \tag{3.6}$$

By Lemma 3.1, $\|\tilde{u}\|_{L^2(\Omega)}$ is uniformly bounded. Therefore, passing to a subsequence if necessary, we may assume that $\tilde{u} \to u^*$ weakly in $L^2(\Omega)$, and $u^* \geq 0$ a.e. in Ω. Dividing (3.6) by α and passing to the limit in (3.6) we have

$$\int_\Omega u^* \nabla\varphi \cdot \nabla m = 0, \tag{3.7}$$

which holds for any $\varphi \in \mathcal{S}$. Since $\mathcal{S}$ is dense in $W^{1,2}(\Omega)$, we see that (3.7) holds for every $\varphi \in W^{1,2}(\Omega)$. In particular, we can choose $\varphi = m$ in (3.7) so that

$$\int_\Omega u^* |\nabla m|^2 \, dx = 0.$$

Hence, $u^*|\nabla m|^2 = 0$ a.e. in Ω. Since the set of critical points of m is of measure zero, we see that $u^* = 0$ a.e. in Ω. Therefore, $u \to 0$ weakly in $L^2(\Omega)$, which implies that $\int_\Omega u \, dx \to 0$ as $\alpha \to \infty$. Hence,

$$\int_\Omega \tilde{u}^2 = \int_\Omega m\tilde{u} \leq \|m\|_{L^\infty(\Omega)} \int_\Omega \tilde{u} \to 0$$

as $\alpha \to \infty$. $\square$

3.3 Concentration at boundary: monotone $m(x)$

In this subsection we restrict ourselves to the case when Ω is an interval and $m(x)$ is monotone. Without loss of generality, we assume that $\Omega = (0,1)$. The goal is to establish

Proof of part (a) of Theorem 1.4. To show that $\tilde{u}'(x) > 0$ in $[0,1]$, we argue by contradiction. If not, since $\tilde{u}_x(0) > 0$ and $\tilde{u}_x(1) > 0$, we have $\tilde{u}_x(\tilde{x}) \le 0$ for some $\tilde{x} \in (0,1)$. Hence, there exists some $x^* \in (0,\tilde{x}]$ such that $\tilde{u}_x(x) > 0$ for every $x \in (0,x^*)$ and $\tilde{u}_x(x^*) = 0$. Integrating the equation of $\tilde{u}$ from 0 to x^*, we have

$$\alpha\tilde{u}(x^*)m_x(x^*) = \int_0^{x^*} \tilde{u}[m-\tilde{u}] \le \int_0^{x^*} \tilde{u}m \le \max_{[0,1]} m \cdot \int_0^{x^*} \tilde{u}. \tag{3.8}$$

Define

$$\kappa = \min_{[0,1]} m_x.$$

By our assumption, $\kappa > 0$. Since $\tilde{u}$ is strictly increasing in $[0,x^*]$, by (3.8) we have

$$\alpha\kappa\tilde{u}(x^*) \le \max_{[0,1]} m \cdot \tilde{u}(x^*).$$

Since $\tilde{u}(x^*) > 0$, we find that $\alpha \le \max_{[0,1]} m/\kappa$. This shows that if $\alpha > \max_{[0,1]} m/\kappa$, then $\tilde{u}'(x) > 0$ in $[0,1]$.

Since $\tilde{u}$ is monotone increasing, it suffices to show that given any $c \in (0,1)$, $\tilde{u}(c) \to 0$ as $\alpha \to \infty$. Since $\tilde{u}$ is monotone increasing in $[c,1]$,

$$\int_c^1 \tilde{u}(x)\,dx \ge (1-c)\tilde{u}(c).$$

As we have shown that $\int_0^1 \tilde{u} \to 0$ as $\alpha \to \infty$, we see that $\tilde{u}(c) \to 0$ as $\alpha \to \infty$.

To complete the proof, we need the following calculus result: if f and g are two monotone functions in $[0,1]$, then

$$\int_0^1 fg \ge \int_0^1 f \cdot \int_0^1 g.$$

Since both m and $\tilde{u}$ are monotone, we have

$$\int_0^1 m \cdot \int_0^1 \tilde{u} \le \int_0^1 m\tilde{u} = \int_0^1 \tilde{u}^2 \le \max_{[0,1]} \tilde{u} \cdot \int_0^1 \tilde{u}.$$

Therefore, $u(1) = \max_{[0,1]} \tilde{u} \ge \int_0^1 m > 0$. $\square$

3.4 Concentration of $\tilde{u}$: general $m(x)$

In this subsection we consider the uniform and pointwise convergence of $\tilde{u}$ and assume that $m(x)$ satisfies (A3). The goal is to prove part (b) of Theorem 1.4.

For any $\delta > 0$, define

$$I_\delta = \{x \in (0,1) : |m'(x)| > \delta\}.$$

Lemma 3.6 *Suppose that* (A1) *and* (A3) *hold. For any $\delta > 0$, there exists some positive constant $C(\delta)$, independent of α, such that $\tilde{u}(x) \le C$ for every $x \in I_\delta$ and every $\alpha \ge 0$.*

Proof. We argue by contradiction. Suppose that the conclusion is false. By Lemma 3.3, $\tilde{u}$ is uniformly bounded for any fixed range of α. Hence, we may assume that there exists $\delta_0 > 0$ such that $\max_{\overline{I}_{\delta_0}} \tilde{u} \to \infty$ as $\alpha \to \infty$. Let $x_\alpha \in \overline{I}_{\delta_0}$ be such that $\tilde{u}(x_\alpha) = \max_{\overline{I}_{\delta_0}} \tilde{u}$. Passing to some sequence if necessary, we may assume that $x_\alpha \to x^* \in \overline{I}_{\delta_0}$ as $\alpha \to \infty$. By (A3), we can write I_{δ_0} as $\cup_{k=1}^K (a_k, b_k)$ for some $K \ge 1$. Hence, $x^* \in [a_i, b_i]$ for some $1 \le i \le K$. By (A3), $x_\alpha \in [a_i, b_i]$ for sufficiently large α, i.e., x_α, x^* belong to the same interval $[a_i, b_i]$.

Set $x = x_\alpha + y/\alpha$, and define

$$w_\alpha(y) = \frac{\tilde{u}(x_\alpha + y/\alpha)}{\tilde{u}(x_\alpha)}.$$

Hence, w_α satisfies $w_\alpha(0) = 1$, $0 < w_\alpha(y) \le 1$, and

$$\frac{d}{dy}\left[\mu \frac{dw_\alpha}{dy} - m'(x_\alpha + y/\alpha) w_\alpha\right] + \frac{1}{\alpha^2} w_\alpha \left[m(x_\alpha + y/\alpha) - \tilde{u}(x_\alpha) w_\alpha\right] = 0$$

in $J_\alpha := (-\alpha(x_\alpha - a_i), \alpha(b_i - x_\alpha))$. As $\alpha \to \infty$, passing to a sequence if necessary, J_α converges to some interval J, where J contains one of the following: $(-\infty, +\infty)$, $[0, +\infty)$, and $(-\infty, 0]$.

Claim. Given any compact subset $\mathcal{K}$ of R^1, $\|w_\alpha\|_{C^2(\mathcal{K})}$ is bounded for sufficiently large α.

To establish our assertion, we first observe that both w_α and $\tilde{u}(x_\alpha)/\alpha$ (Lemma 3.3) are uniformly bounded for large α. Integrating the equation of $\tilde{u}$ from $x = 0$ to $x = x_\alpha$, we have

$$\mu \tilde{u}'(x_\alpha) - \alpha m'(x_\alpha) \tilde{u}(x_\alpha) + \int_0^{x_\alpha} \tilde{u}(m - \tilde{u}) = 0.$$

Hence, $\tilde{u}'(x_\alpha)/(\alpha \tilde{u}(x_\alpha))$ is uniformly bounded for large α. Note that here it suffices to assume that $\tilde{u}(x_\alpha)$ is uniformly bounded below by some positive constant. This implies that $w_\alpha'(0)$ is uniformly bounded since $w_\alpha'(0) = \tilde{u}'(x_\alpha)/(\alpha \tilde{u}(x_\alpha))$. Now integrating the equation of w_α from 0 to y, we find that

$$\begin{aligned} &\mu w_\alpha'(y) - m'(x_\alpha + y/\alpha) w_\alpha(y) - \mu w_\alpha'(0) + m'(x_\alpha) w_\alpha(0) \\ &+ \frac{1}{\alpha^2} \int_0^y w_\alpha [m(x_\alpha + y/\alpha) - \tilde{u}(x_\alpha) w_\alpha] \, dy = 0. \end{aligned} \tag{3.9}$$

Therefore, $\|w_\alpha\|_{C^1(\mathcal{K})}$ is uniformly bounded for large α. By the equation of w_α, we see that $\|w_\alpha\|_{C^2(\mathcal{K})}$ is uniformly bounded. This proves our assertion.

By our assertion and a standard diagonal process, passing to a sequence if necessary, we see that $w_\alpha \to w^*$ in $C^1(\mathcal{K})$, where $\mathcal{K}$ is any compact subset of J. By the equation of w_α, $w_\alpha \to w^*$ in $C^2(\mathcal{K})$. Hence, w^* satisfies $w^*(0) = 1$ and $0 \le w^* \le 1$.

By Lemma 3.4, we have

$$\int_0^1 \tilde{u}(x)[m'(x)]^2\,dx \le \frac{C}{\alpha}.$$

Since $|m'| \ge \delta_0$ in $(a_i, b_i) \subset I_{\delta_0}$, we have

$$\int_{a_i}^{b_i} \tilde{u}(x)\,dx \le \frac{C}{\delta_0^2 \alpha}.$$

By the change of variable $x = x_\alpha + y/\alpha$ and the definition of w_α, we obtain

$$\int_{J_\alpha} w_\alpha(y)\,dy \le \frac{C}{\delta_0^2 \tilde{u}(x_\alpha)}.$$

In particular,

$$\int_{J_\alpha \cap (-1,1)} w_\alpha(y)\,dy \le \frac{C}{\delta_0^2 \tilde{u}(x_\alpha)}. \tag{3.10}$$

Passing to the limit in (3.10), by $\tilde{u}(x_\alpha) \to \infty$ we have

$$\int_{J \cap (-1,1)} w^*(y)\,dy \le 0.$$

This implies that $w^* \equiv 0$ in $J \cap (-1,1)$, which contradicts $w^*(0) = 1$ since $0 \in J \cap (-1,1)$. □

Theorem 3.7 *Suppose that* (A1) *and* (A3) *hold. For any $\delta > 0$, $\tilde{u} \to 0$ uniformly in I_δ as $\alpha \to \infty$. In particular, $\tilde{u}(x) \to 0$ for every $x \in [0,1] \setminus \{x_1, ..., x_k\}$ as $\alpha \to \infty$.*

Proof. We argue by contradiction. Passing to a sequence if necessary, we assume that there exist $\delta_0 > 0$ and $\eta > 0$ such that $\tilde{u}(x) \ge \eta$ for some $x^\alpha \in I_{\delta_0}$ and sufficiently large α. Choose $x_\alpha \in \overline{I}_{\delta_0}$ such that $\tilde{u}(x_\alpha) = \max_{\overline{I}_{\delta_0}} \tilde{u} \ge \eta$. Set $x = x_\alpha + y/\alpha$ and define $w_\alpha = \tilde{u}(x_\alpha + y/\alpha)$. Hence, $w_\alpha(0) \ge \eta$. Passing to a subsequence if necessary, we may assume that $x_\alpha \to x^* \in \overline{I}_{\delta_0}$ as $\alpha \to \infty$. By (A3), we can write I_{δ_0} as $\cup_{k=1}^K (a_k, b_k)$ for some $K \ge 1$. Hence, $x^* \in [a_i, b_i]$ for some $1 \le i \le K$. By (A3), we may assume that $x_\alpha \in [a_i, b_i]$ for sufficiently large α. By assumption $m'(0) \ge 0 \ge m'(1)$, so there are only three possibilities: $0 < a_i < b_i < 1$, $0 = a_i < b_i < 1$, and $0 < a_i < b_i = 1$.

We first consider the case $0 < a_i < b_i < 1$. For this case, we can find some interval $(c_i, d_i) \subset I_{\delta_0/2}$ such that $[a_i, b_i] \subset (c_i, d_i)$. Then w_α satisfies

$$\frac{d}{dy}\left[\mu \frac{dw_\alpha}{dy} - m'(x_\alpha + y/\alpha) w_\alpha\right] + \frac{1}{\alpha^2} w_\alpha \left[m(x_\alpha + y/\alpha) - w_\alpha\right] = 0$$

in $J_\alpha := (-\alpha(x_\alpha - c_i), \alpha(d_i - x_\alpha))$. Since $x_\alpha \in [a_i, b_i]$, we see that J_α converges to $(-\infty, +\infty)$ as $\alpha \to \infty$. By Lemma 3.6, w_α is uniformly bounded in J_α. Similarly as in the proof of Lemma 3.6, passing to some sequence if necessary, we may assume that

$w_\alpha \to w^*$ in $C^2(\mathcal{K})$, where $\mathcal{K}$ is any compact subset of $(-\infty, +\infty)$. Hence, w^* satisfies $w^*(0) \geq \eta$, $0 \leq w^*(y) \leq C$ in $(-\infty, +\infty)$, and

$$\mu \frac{d^2 w^*}{dy^2} - m'(x^*) \frac{dw^*}{dy} = 0 \qquad \text{in } (-\infty, +\infty).$$

Hence, $w^* = c_1 + c_2 e^{(m'(x^*)/\mu)y}$ for some constants c_1 and c_2. Since w^* is bounded in $(-\infty, +\infty)$, we see that $c_2 = 0$. This together with $w^*(0) \geq \eta$ implies that $w^* \equiv w^*(0)$ in $(-\infty, +\infty)$.

By Lemma 3.4, we have

$$\int_{c_i}^{d_i} \tilde{u}(x)[m'(x)]^2 \, dx \leq \frac{C}{\alpha}.$$

Since $|m'| \geq \delta_0/2$ in $(c_i, d_i) \subset I_{\delta_0/2}$, by the change of variable $x = x_\alpha + y/\alpha$ and the definition of w_α, we obtain

$$\int_{J_\alpha} w_\alpha(y) \, dy \leq \frac{4C}{\delta_0^2}.$$

For any $L > 0$, $[-L, L] \subset J_\alpha$ for sufficiently large α. Hence,

$$\int_{-L}^{L} w_\alpha(y) \, dy \leq \frac{4C}{\delta_0^2}.$$

Passing to the limit we find

$$\int_{-L}^{L} w^*(y) \, dy \leq \frac{4C}{\delta_0^2},$$

i.e., $2L\eta \leq (4C)/(\delta_0^2)$ since $w^* \geq \eta$. This is a contradiction since $L > 0$ is arbitrary.

Next we consider the case $a_i = 0$ and $b_i < 1$. For this case, if $x^* > 0$, then we can use the same proof as above to reach a contradiction. (The main point is that for this case we also have $J_\alpha \to (-\infty, +\infty)$ as $\alpha \to \infty$, which again implies that w^* is equal to some positive constant) It remains to consider the case $x^* = 0$. Since $|m'(x_\alpha)| \geq \delta_0 > 0$ and $x_\alpha \to x^* = 0$, we see that $|m'(0)| \geq \delta_0$. Since we assume that $m'(0) \geq 0$, we have $m'(0) > 0$. By the same argument as before, we can assume that $w_\alpha \to w^*$ as $\alpha \to \infty$, $w^*(0) \geq \eta$, $0 \leq w^* \leq C$, and w^* satisfies

$$\mu \frac{d^2 w^*}{dy^2} - m'(0) \frac{dw^*}{dy} = 0$$

in some interval J which contains $[0, +\infty)$. Hence, $w^* = c_1 + c_2 e^{(m'(0)/\mu)y}$ in $[0, +\infty)$. Since $m'(0) > 0$, $w^*(0) \geq \eta$, and w^* is bounded, the only possibility is that $w^* \equiv w^*(0) \geq \eta$ in $[0, \infty)$. Then, as in the case $0 < a_i < b_i < 1$ (with $[-L, L]$ being replaced by $[0, L]$), as in previous case we can apply Lemma 3.4 to reach a contradiction.

The case $a_i > 0$ and $b_i = 1$ can be similarly treated. This completes the proof. □

4 Instability of semi-trivial states for $\alpha \gg 1$

In this section we study the stability of the two semi-trivial states $(\tilde{u}, 0)$ and $(0, \theta(\cdot, \nu))$ and establish Theorem 1.5.

4.1 Instability of $(\tilde{u}, 0)$

Theorem 4.1 *Suppose that* (A1) *and* (A2) *hold. Then for fixed $\mu > 0$, there exists some positive constant $\alpha_1 = \alpha_1(\mu, \Omega)$ such that if $\alpha \geq \alpha_1$, $(\tilde{u}, 0)$ is unstable for every $\nu > 0$.*

Proof. It suffices to show that the least eigenvalue σ_1 for the problem

$$\nu \Delta \psi + (m - \tilde{u})\psi = -\sigma \psi \quad \text{in } \Omega, \quad \left.\frac{\partial \psi}{\partial n}\right|_{\partial\Omega} = 0$$

is negative for $\alpha \gg 1$. Let $\psi_1 > 0$ in Ω be an eigenfunction associated with σ_1. By the maximum principle, $\psi_1 > 0$ in $\overline{\Omega}$. Dividing the equation of ψ_1 by ψ_1, integrating in Ω, we obtain

$$\sigma_1 |\Omega| = -\nu \int_\Omega \frac{|\nabla \psi_1|^2}{\psi_1^2} + \int_\Omega (\tilde{u} - m) \leq \int_\Omega (\tilde{u} - m).$$

By Theorem 3.5, $\int_\Omega \tilde{u} \to 0$ as $\alpha \to \infty$. Since $\int_\Omega m > 0$, we find that $\sigma_1 < 0$ for $\alpha \gg 1$ and every $\nu > 0$. $\square$

4.2 Instability of $(0, \theta(\cdot, \nu))$

For simplicity, in this and next subsection we denote $\theta(\cdot, \nu)$ by $\tilde{v}$. The goal is to study the stability of $(0, \tilde{v})$ for various ranges of values of α. We first establish some *a priori* estimates of $\tilde{v}$.

Lemma 4.2 *For every $\nu > 0$, we have $\max_{\overline{\Omega}} \tilde{v} < \max_{\overline{\Omega}} m$.*

Proof. By the maximum principle, $\|\tilde{v}\|_{L^\infty(\Omega)} \leq \max_{\overline{\Omega}} m$. Set $v_1 = \max_{\overline{\Omega}} m - \tilde{v}$. Hence, $v_1 \geq 0$ in Ω. Since $\tilde{v}$ is a non-constant function (as m is not-constant), $v_1 \not\equiv 0$. By the equation of $\tilde{v}$, we see that v_1 satisfies

$$-\nu \Delta v_1 + v_1(\tilde{v} + \max_{\overline{\Omega}} m - m) = \max_{\overline{\Omega}} m (\max_{\overline{\Omega}} m - m) \geq 0$$

in Ω and $\partial v_1 / \partial n = 0$ on $\partial\Omega$. Since $v_1 \geq 0$ and $v_1 \not\equiv 0$, by the maximum principle we have $v_1 > 0$ in $\overline{\Omega}$. This completes the proof. $\square$

Lemma 4.3 *For any $\eta > 0$, there exists $\delta = \delta(\eta, \Omega) > 0$ such that for every $\nu \geq \eta$,*

$$\max_{\overline{\Omega}} \tilde{v} \leq \max_{\overline{\Omega}} m - \delta.$$

Proof. We argue by contradiction. Suppose that the conclusion is false. By Lemma 4.2 we may assume that there exists $\eta_0 > 0$, $\nu_i \geq \eta_0$ and $v_i = \tilde{v}(\cdot, \nu_i)$ satisfying $\max_{\overline{\Omega}} v_i \to \max_{\overline{\Omega}} m$ as $i \to \infty$. By standard elliptic regularity, there exists $\gamma \in (0,1)$ such that $\|v_i\|_{C^{2,\gamma}(\overline{\Omega})}$ is uniformly bounded. Passing to a sequence if necessary, we may assume that either $\nu_i \to \hat{\nu}$ for some $\hat{\nu} > 0$ or $\nu_i \to \infty$, and $v_i \to v^*$ in $C^2(\overline{\Omega})$. In particular, $v^* \geq 0$ and satisfies $\max_{\overline{\Omega}} v^* = \max_{\overline{\Omega}} m$. If $\nu_i \to \hat{\nu}$, then $v^* = \tilde{v}(\cdot, \hat{\nu})$. This implies that $\max_{\overline{\Omega}} \tilde{v}(\cdot, \hat{\nu}) = \max_{\overline{\Omega}} m$, which contradicts Lemma 4.2. If $\nu_i \to \infty$, we see that $v^* \equiv \int_\Omega m/|\Omega|$. Hence, $\max_{\overline{\Omega}} m = \int_\Omega m/|\Omega|$, which implies that m is a constant. This contradiction completes the proof. $\square$

Theorem 4.4 *Suppose that* (A1) *and* (A4) *hold. For every $\mu > 0$ and $\eta > 0$, there exists some positive constant $\alpha_2 = \alpha_2(\mu, \eta, \Omega)$ such that if $\alpha \geq \alpha_2$, then $(0, \tilde{v})$ is unstable for every $\nu \geq \eta$.*

Proof. It suffices to show that the eigenvalue problem

$$\begin{aligned} &\nabla \cdot [\mu \nabla \varphi - \alpha \varphi \nabla m] + (m - \tilde{v})\varphi = -\sigma \varphi \quad \text{in } \Omega, \\ &[\mu \nabla \varphi - \alpha \varphi \nabla m] \cdot n = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{4.1}$$

has an eigenvalue with negative real part. Set $\psi = e^{-(\alpha/\mu)m}\varphi$. Then ψ satisfies

$$\begin{aligned} &\mu \nabla \cdot \left(e^{(\alpha/\mu)m} \nabla \psi\right) + (m - \tilde{v}) e^{(\alpha/\mu)m} \psi = -\sigma e^{(\alpha/\mu)m} \psi \quad \text{in } \Omega, \\ &\frac{\partial \psi}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.2}$$

To show that the least eigenvalue of (4.2) is negative, it suffices to find Ψ such that

$$\mu \int_\Omega e^{(\alpha/\mu)m} |\nabla \Psi|^2 < \int_\Omega (m - \tilde{v}) e^{(\alpha/\mu)m} \Psi^2. \tag{4.3}$$

By Lemma 4.3, there exists $x_0 \in \overline{\Omega}$ such that $m(x_0) = \max_{\overline{\Omega}} m$ and $m(x_0) - \tilde{v}(x_0) \geq \delta$ for every $\nu \geq \eta$. Standard elliptic regularity and the Sobolev embedding theorem imply that there exists some positive constant $C_1 = C_1(\eta, \Omega)$ such that if $\nu \geq \eta$, then $\|\nabla \tilde{v}\|_{L^\infty} \leq C_1$. Here and below numbers C_i always denote some positive constants depending only on η and Ω. Hence, there exists $R_1 = R_1(\eta, \Omega)$ small such that $m - \tilde{v} \geq \delta/2$ in $B_{R_1}(x_0) \cap \Omega$ for every $\nu \geq \eta$. For brevity, we shall write $B_R(x_0)$ as B_R for any $R > 0$.

For $R_1, R_2 > 0$, define

$$\begin{aligned} M_1 &= \max_{(B_{R_1} \setminus B_{R_1/2}) \cap \Omega} m, \\ M_2 &= \min_{B_{R_2} \cap \Omega} m. \end{aligned} \tag{4.4}$$

By assumption (A4), we can choose $R_2 \leq R_1/2$ small enough such that

$$M_2 \geq [M_1 + m(x_0)]/2. \tag{4.5}$$

Choose $\Psi \in C^1(\overline{\Omega})$ such that

$$\Psi = \begin{cases} 1 & \text{in } B_{R_1/2} \cap \Omega, \\ \in [0,1] & \text{in } (B_{R_1} \setminus B_{R_1/2}) \cap \Omega, \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

In particular, $|\nabla \Psi|_{L^\infty} \leq C_3$. Then

$$\begin{aligned} \mu \int_\Omega e^{(\alpha/\mu)m} |\nabla \Psi|^2 &= \mu \int_{(B_{R_1} \setminus B_{R_1/2}) \cap \Omega} e^{(\alpha/\mu)m} |\nabla \Psi|^2 \\ &\leq C_4 \int_{(B_{R_1} \setminus B_{R_1/2}) \cap \Omega} e^{(\alpha/\mu)m} \\ &\leq C_5 e^{(\alpha/\mu)M_1}, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \int_\Omega (m - \tilde{v}) e^{(\alpha/\mu)m} \Psi^2 &= \int_{B_{R_1} \cap \Omega} (m - \tilde{v}) e^{(\alpha/\mu)m} \Psi^2 \\ &\geq \int_{B_{R_2} \cap \Omega} (m - \tilde{v}) e^{(\alpha/\mu)m} \\ &\geq e^{(\alpha/\mu)M_2} \int_{B_{R_2} \cap \Omega} (m - \tilde{v}) \\ &\geq C_6 e^{(\alpha/\mu)M_2}. \end{aligned} \tag{4.8}$$

By (4.5), (4.7), and (4.8), choosing α sufficiently large we see that (4.3) holds. This completes the proof. $\square$

Proof of Theorem 1.5. Parts (a) and (b) are given by Theorems 4.1 and 4.4. Since (1.3)-(1.4) is a strongly monotone system (Lemma 2.2), part (c) follows from (a), (b) and theory for monotone systems (see, e.g., Corollary 7.6 and Theorem 10.2 of [13]). To prove part (d), let (u_α, v_α) be a coexistence state of (1.3)-(1.4). Then u_α satisfies

$$\begin{aligned} &\nabla \cdot [\mu \nabla u_\alpha - \alpha u_\alpha \nabla m] + (m - u_\alpha) u_\alpha > 0 \quad \text{in } \Omega, \\ &\mu \frac{\partial u_\alpha}{\partial n} - \alpha u_\alpha \frac{\partial m}{\partial n} = 0 \quad \text{in } \partial\Omega, \end{aligned} \tag{4.9}$$

i.e., u_α is a subsolution of (1.5). By the standard supersolution and subsolution method we have $u_\alpha \leq \tilde{u}$ in $\overline{\Omega}$. This together with part (a) and (b) of Theorem 1.3 imply that $\|u_\alpha\|_{L^2(\Omega)} \to 0$ and $u_\alpha \to 0$ pointwise in $[0,1] \setminus \{x_1, ..., x_k\}$ as $\alpha \to \infty$. By standard elliptic regularity [10] and $\|u_\alpha\|_{L^2(\Omega)} \to 0$, we have $v_\alpha \to \theta(\cdot, \nu)$ in $W^{2,2}(\Omega)$ as $\alpha \to \infty$. $\square$

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