ON THE EFFECTS OF MIGRATION AND SPATIAL HETEROGENEITY ON SINGLE AND MULTIPLE SPECIES

YUAN LOU

Department of Mathematics
The Ohio State University
Columbus, Ohio 43210, USA

MSC: 35B40; 35K57; 92D10; 92D25

Keywords: Reaction-diffusion; Competing species; Spatial heterogeneity; Population size; Invasion

E-mail address: lou@math.ohio-state.edu

Abstract

We first investigate in a logistic model the effects of migration and spatial heterogeneity of the environment on the total population size at equilibrium of a single species. Our study shows that (i) the total population size is maximized at some intermediate migration rate, and hence is a non-monotone function of the migration rate; (ii) heterogeneity of the environment increases the population size. In the second part of this paper, these findings are applied to ecological invasions. For a two-species Lotka-Volterra competition model with migration, we show that (i) without migration, the invading species eliminates the resident species at every point of the habitat, whereas when migration is present, for certain ranges of migration rates the invader may be eliminated when it is rare; and (ii) without migration, the two species can coexist at every point of the habitat, whereas when migration is present, for some ranges of migration rates one of the species is extinguished for all initial conditions.

1. Introduction

In the first part of this paper, we study the effects of migration and spatial heterogeneity of the environment on the total population size of a single species. More precisely, we consider the semilinear elliptic equation

$$\mu\Delta\theta + \theta\big[m(x) - \theta\big] = 0 \quad \text{in } \Omega, \qquad \theta > 0 \quad \text{in } \Omega, \tag{1.1a}$$

$$\frac{\partial\theta}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{1.1b}$$

where the migration rate μ is assumed to be a positive constant, the function $\theta = \theta(x,\mu)$ represents the density of the species at location x, and $m(x)$ denotes its local intrinsic growth rate. The habitat of the species Ω is a bounded region in R^N with smooth boundary $\partial\Omega$, and n is the outward unit normal vector on $\partial\Omega$. The zero-flux boundary condition (1.1b) means that no individuals cross the boundary of the habitat.

Though some of the analyses here cover the case $\int_\Omega m(x)\,dx \le 0$, throughout this paper, for the sake of clarity, we posit that

(**A**) $m(x)$ is non-constant, bounded and measurable, and $\int_\Omega m(x)\,dx > 0$.

For solutions of (1.1), we present the following results.

Theorem 1.1. *Suppose that assumption* (A) *holds.*

(a) *For every $\mu > 0$, the problem* (1.1) *has a unique solution $\theta(x,\mu)$ such that $\theta \in W^{2,p}(\Omega)$ for every $p \ge 1$.*

(b) *As $\mu \to 0+$, the solution $\theta(x,\mu) \to m_+(x)$ in $L^p(\Omega)$ for every $p \ge 1$, where $m_+(x) = \sup\{m(x), 0\}$; as $\mu \to \infty$, the solution $\theta(x,\mu) \to \frac{1}{|\Omega|}\int_\Omega m(x)\,dx$ in $W^{2,p}(\Omega)$ for every $p \ge 1$.*

(c) *If $m(x)$ is Hölder continuous in $\overline{\Omega}$, then $\theta \in C^2(\bar{\Omega})$. Moreover, $\theta(x,\mu) \to m_+(x)$ in $L^\infty(\Omega)$ as $\mu \to 0$, and $\theta(x,\mu) \to \frac{1}{|\Omega|}\int_\Omega m(x)\,dx$ in $C^2(\overline{\Omega})$ as $\mu \to \infty$.*

We refer the proofs of parts (a) and (c) to [5] and the references therein. Since we can not locate the proof of the first part of (b) in the literature, we prove it in the beginning of Section 2.

In view of part (b) of Theorem 1.1, it is natural to introduce the function

$$F(\mu) \equiv \begin{cases} \displaystyle\int_\Omega m_+(x)\,dx, & \mu = 0, \\ \displaystyle\int_\Omega \theta(x,\mu)dx, & \mu > 0, \\ \displaystyle\int_\Omega m(x)\,dx, & \mu = \infty, \end{cases} \tag{1.2}$$

which can be interpreted as the total population size of the species. By assumption (A) and part (b) of Theorem 1.1, F is a continuous, positive function in $[0,\infty]$.

If the spatial environment is homogeneous, i.e., $m(x)$ is equal to some positive constant $\overline{m}$, then $\theta(x,\mu) \equiv \overline{m}$ is the unique positive solution of (1.1) for every $\mu > 0$. In this case, the total population size of the species is given by $F(\mu) = |\Omega|\overline{m}$, which is independent of μ. However, if the spatial environment is heterogeneous, i.e., $m(x)$ is a non-constant function, the story changes dramatically, as shown in

Theorem 1.2. *Suppose that assumption* (A) *holds.*

(a) *The function $F(\mu)$ satisfies $F(\mu) > F(\infty)$ for every $\mu \in (0,\infty)$.*

(b) *If $m(x) \geq 0$ in Ω, then for every $\mu \in (0,\infty)$, the function $F(\mu)$ satisfies*

$$F(0) = F(\infty) < F(\mu). \tag{1.3}$$

Remark 1.3. Part (a) of Theorem 1.2 implies that spatial heterogeneity increases the population size of species. To make this assertion precise, set $\overline{m} = \int_\Omega m(x)\,dx/|\Omega|$, and write $F = F(\mu, m)$ instead of $F(\mu)$ to indicate the dependence of F on the function m. Part (a) implies that $F(\mu, m) > F(\mu, \overline{m})$ for every $\mu > 0$. In other words, given any $\mu > 0$ and any function g with $\int_\Omega g(x)\,dx = 0$ and $g \not\equiv 0$, we have $F(\mu, \overline{m} + \lambda g) > F(\mu, \overline{m})$ for every $\lambda \neq 0$. Hence, with the migration rate fixed, the population size $F(\mu, \overline{m} + \lambda g)$, as a function of λ, attains a strict absolute minimum at $\lambda = 0$.

Remark 1.4. Part (b) of Theorem 1.2 implies that when $m(x)$ is non-negative, the total population size is minimized at $\mu = 0$ and $\mu = \infty$, and maximized at some intermediate value μ^*. The value of μ^* is determined by the habitat Ω and $m(x)$. For typical choices of $m(x)$, numerical computations suggest that $10^{-3} \leq \mu^* \leq 10^{-1}$.

From the biological point of view, the region $\{x \in \Omega : m(x) > 0\}$ acts like a source for species, whereas $\{x \in \Omega : m(x) < 0\}$ can be thought as a sink. The case when the growth rate $m(x)$ changes sign seems to be as interesting as the case when m is non-negative. If the function $m(x)$ changes sign, then (1.2) and part (a) of Theorem 1.2 show that $F(\mu) > F(\infty)$ for every $\mu \in [0, \infty)$. Hence, F is minimized at $\mu = \infty$, but is no longer minimized at $\mu = 0$. However, when Ω is an interval, the maximum of F is still attained at some intermediate migration rate, as shown by

Theorem 1.5. *Suppose that $\Omega = (0,1)$, $m \in C^2[0,1]$, m changes sign, and $m(x) = 0$ has only nondegenerate roots in $[0,1]$. Then there exist positive constants μ_0 and c_0 such that $F(\mu) - F(0) \geq c_0 \mu^{\frac{2}{3}}$ for every $\mu \in (0, \mu_0)$. Thus,*

$$F(\infty) < F(0) < \sup_{0<\mu<\infty} F(\mu). \tag{1.4}$$

Theorems 1.2 and 1.5 suggest that the total population size of species is usually maximized at some intermediate migration rate, and it turns out that this fact has interesting applications to multiple species in the context of ecological invasions. In the second part of this paper, we apply Theorems 1.2 and 1.5 to study the Lotka-Volterra competition-diffusion model

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \mu \Delta u + u\,[m(x) - u - bv] \qquad \text{in } \ \Omega \times (0,\infty),\\
&\frac{\partial v}{\partial t} = \nu \Delta v + v\,[m(x) - cu - v] \qquad \text{in } \ \Omega \times (0,\infty),\\
&\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \qquad \text{on } \ \partial\Omega \times (0,\infty),\\
&u(x,0) = u_0(x), \quad v(x,0) = v_0(x),
\end{aligned} \tag{1.5}$$

where $u(x,t)$ and $v(x,t)$ represent the population densities of competing species 1 and 2 with respective migration rates μ and ν, the function $m(x)$ represents their common intrinsic growth rate, and b and c are inter-specific competition coefficients. We shall assume that μ, ν, b, and c are positive constants, the growth rate $m(x)$ satisfies (A), and u_0 and v_0 are non-negative functions that are not identically equal to zero.

For the last two decades there has been tremendous interest, by both mathematicians and ecologists, in two-species Lotka-Volterra competition models with spatially

heterogeneous interactions, see [2-8, 11-15, 18, 20, 22-27, 29, 30, 32, 33, 35, 37] and references therein.

To motivate our discussion, we assume for the moment that $0 < b, c < 1$. If $m(x) \equiv \overline{m}$ for some positive constant $\overline{m}$, then every solution (u, v) of (1.5) converges to $(\frac{1-b}{1-bc}\overline{m}, \frac{1-c}{1-bc}\overline{m})$ for all diffusion rates μ, ν, and arbitrary initial data. Hence, it is natural to investigate the asymptotic behavior of solutions of (1.5) when $m(x)$ is a non-constant positive function and $0 < b, c < 1$.

If $\mu = \nu = 0$ in (1.5), we can regard (1.5) as a system of two ordinary differential equations, solutions of which converge to $\left(\frac{1-b}{1-bc}m(x), \frac{1-c}{1-bc}m(x)\right)$ for every $x \in \Omega$. In other words, the kinetic system of (1.5) has a unique, globally asymptotically stable positive equilibrium (depending on x). It is shown in [25] that if μ and ν are sufficiently small, the function $m(x)$ is positive, and $0 < b, c < 1$, then (1.5) has a unique, globally asymptotic stable positive steady state (u^*, v^*). Moreover, (u^*, v^*) converges to $\left(\frac{1-b}{1-bc}m(x), \frac{1-c}{1-bc}m(x)\right)$ in $L^\infty(\Omega)$ as $\mu \to 0$ and $\nu \to 0$. Loosely speaking, when both migration rates μ and ν are small, the dynamics of (1.5) behaves rather similarly to that of (1.5) with $\mu = \nu = 0$.

When both μ and ν are sufficiently large, it is not difficult to see that (1.5) again has a unique, globally asymptotically stable positive steady state (u^{**}, v^{**}). Moreover, (u^{**}, v^{**}) converges to $\left(\frac{1-b}{1-bc}\frac{1}{|\Omega|}\int_\Omega m(x)\,dx, \frac{1-c}{1-bc}\frac{1}{|\Omega|}\int_\Omega m(x)\,dx\right)$ as $\mu \to \infty$ and $\nu \to \infty$. In fact, when μ and ν are sufficiently large, the dynamics of (1.5) can be approximated by that of the corresponding "spatially averaged" ODE. Consult [1, 9, 10, 17] and references therein for the connection between the dynamics of PDEs with large diffusion and that of corresponding "spatially averaged" ODEs.

Hence, it seems reasonable to expect that for other ranges of migration rates, the dynamics of (1.5) still should be well behaved; e.g., the two competing species can coexist, as in the case when μ and ν are both small or both large, or in the case when $m(x)$ is equal to some positive constant. However, as we shall see later, given any non-constant positive function $m(x)$, there exists a set of parameters $b, c \in (0, 1)$ and $\mu, \nu > 0$ such that one of the semi-trivial steady states of (1.5) is the global attractor of (1.5). Therefore, the joint action of spatial heterogeneity and diffusion can drive one of the species to extinction. Such "diffusion-driven extinction" phenomena

have been studied in [28] and [31], where it was proved that there exist some initial data such that without diffusion species 1 drives species 2 to extinction, whereas the opposite holds for certain migration rates. In this paper, the spatial heterogeneity is incorporated in the nonlinearity, and the extinction result here holds for arbitrary initial conditions.

We start by studying the joint effects of migration and spatial heterogeneity on the invasion of the species 2 when it is rare. The case of species 1 is similar. Mathematically, this is equivalent to studying the stability of the semi-trivial steady state $(\theta, 0)$ of (1.5), where $\theta = \theta(x, \mu)$ is the unique positive solution of (1.1). In this paper, unless otherwise specified, by stability we mean linear stability. We first recall the following result in [11].

Theorem 1.6. *Suppose that $c = 1$ and assumption* (A) *holds.*

(a) *The steady state $(\theta, 0)$ is stable when $\mu < \nu$, and unstable when $\mu > \nu$.*

(b) *If also $b = 1$, then $(\theta, 0)$ is globally asymptotically stable when $\mu < \nu$.*

In the rest of this paper, we are primarily interested in the case $0 < c < 1$. As an application of Theorem 1.2, the stability of $(\theta, 0)$ when $c \in (0, 1)$ can be described by

Theorem 1.7. *If assumption* (A) *holds and $m(x)$ is non-negative, then there exists some constant $c_* = c_*(m, \Omega) \in (0, 1)$ such that the following results hold.*

(a) *For every $c \in (0, c_*)$, the steady state $(\theta, 0)$ is unstable when $\mu > 0$ and $\nu > 0$.*

(b) *For every $c \in (c_*, 1)$, there exists $\bar{\nu} = \bar{\nu}(c, m, \Omega) > 0$ such that* (i) *for every $\nu \in (0, \overline{\nu})$, the steady state $(\theta, 0)$ is unstable when $\mu > 0$;* (ii) *for every $\nu > \bar{\nu}$, the steady state $(\theta, 0)$ changes stability at least twice as μ increases from 0 to ν.*

Remark 1.8. In Sect. 4 we shall give a detailed description of the region in the $\mu\nu$-plane where $(\theta, 0)$ is stable. The most interesting region is where $c_* < c < 1$ and $\nu > \bar{\nu}$, where we have the following.

(a) If $b > 1$, it is well known that without migration, species 2 always drives species 1 to extinction. However, Theorem 1.7 shows that with migration, for some ranges of migration rates, species 2 may fail to invade when rare.

(b) If $b < 1$, it is well known that, without migration, species 1 always coexists with species 2. Surprisingly, as shown in Theorem 1.9, for certain migration rates, species 2 is able to drive species 1 to extinction for arbitrary initial conditions.

For every $c > 0$, define

$$\Sigma_c = \big\{(\mu, \nu) \in (0, \infty) \times (0, \infty) : (\theta, 0) \text{ is linearly stable}\big\}. \tag{1.6}$$

By Theorem 1.7, the set Σ_c is non-empty for every $c \in (c_*, 1)$.

Theorem 1.9. *If assumption* (A) *holds and $m(x)$ is non-negative, then for every $c \in (c_*, 1)$, there exists $b_* = b_*(c, \Omega, m) \in (0, 1]$ such that if $b \in (0, b_*)$ and $(\mu, \nu) \in \Sigma_c$, then $(\theta, 0)$ is globally asymptotically stable.*

We conjecture that Theorem 1.9 holds with $b_* = 1$.

For the case when $m(x)$ changes sign, the stability of $(\theta, 0)$ is somewhat different. Define

$$\rho = \frac{\int_\Omega m(x)\, dx}{\int_\Omega m_+(x)\, dx}. \tag{1.7}$$

Theorem 1.10. *Suppose that $m(x)$ satisfies the assumptions in Theorem* 1.5. *Then there exists some constant $c_* = c_*(m, \Omega) \in (0, \rho)$ such that*

(a) *for every $c \in (0, c_*)$ and every $\mu > 0$ and $\nu > 0$, the steady state $(\theta, 0)$ is unstable;*

(b) *for every $c \in (c_*, \rho)$, there exists $\bar{\nu} = \bar{\nu}(c, \Omega) > 0$ such that $(\theta, 0)$ is unstable when $\mu > 0$ and $\nu \in (0, \bar{\nu})$, and for every fixed $\nu > \bar{\nu}$, the steady state $(\theta, 0)$ changes stability at least twice as μ increases from* 0 *to ν;*

(c) *for every $c \in [\rho, 1)$, there exists $\hat{\nu} = \hat{\nu}(c, \Omega) > 0$ such that $(\theta, 0)$ is unstable when $\mu > 0$ and $\nu \in (0, \hat{\nu})$, and for every fixed $\nu > \hat{\nu}$, the steady state $(\theta, 0)$ changes stability at least once as μ increases from* 0 *to ν.*

This paper is organized as follows: In Section 2 we establish Theorem 1.2, and Section 3 is devoted to the proof of Theorem 1.5. Theorems 1.7 and 1.10 are proved in Section 1.4. Finally, in Section 1.5 we study the global dynamics of (1.5) and establish Theorem 1.9.

2. The population size of a single species: non-negative $m(x)$

In this section, we study the population size of a single species, and our main goal is to establish Theorem 1.2. Let $\theta = \theta(x, \mu)$ be the solution of (1.1). If m is merely bounded measurable, the convergence of $\theta \to m_+$ as $\mu \to 0$ can not be in L^∞ norm. It turns out that the convergence holds in L^p norm for every $p \geq 1$, as shown in

Lemma 2.1. *Suppose that assumption* (A) *holds. Then for every* $p \geq 1$, $\|\theta - m_+\|_{L^p(\Omega)} \to 0$ *as* $\mu \to 0$.

Proof. By the maximum principle [36], we have $\|\theta\|_{L^\infty(\Omega)} \leq \|m\|_{L^\infty(\Omega)}$ for every $\mu > 0$. Hence, it suffices to show that $\|\theta - m_+\|_{L^1(\Omega)} \to 0$ as $\mu \to 0$. For every $\epsilon > 0$, using the mollifier of m, we can find $\tilde{m} \in C^1(\overline{\Omega})$ such that

$$\|\tilde{m}\|_{L^\infty(\Omega)} \leq \|m\|_{L^\infty(\Omega)}, \qquad \|\tilde{m} - m\|_{L^1(\Omega)} \leq \epsilon. \tag{2.1}$$

By (2.1) and assumption (A), we can choose ϵ so small that $\int_\Omega \tilde{m}\, dx > 0$. Hence, for every $\mu > 0$, the following equation

$$\mu \Delta \tilde{\theta} + \tilde{\theta}\left[\tilde{m}(x) - \tilde{\theta}\right] = 0 \qquad \text{in } \Omega, \qquad \frac{\partial \tilde{\theta}}{\partial n}\Big|_{\partial\Omega} = 0 \tag{2.2}$$

has a unique positive classical solution. It is well known that $\tilde{\theta} \to \tilde{m}_+$ uniformly in $\bar{\Omega}$ as $\mu \to 0+$. By the maximum principle, we have

$$\|\tilde{\theta}\|_{L^\infty} \leq \|\tilde{m}\|_{L^\infty} \leq \|m\|_{L^\infty}. \tag{2.3}$$

Claim. There exists some positive constant C_1, independent of ϵ and μ, such that

$$\|\tilde{\theta} - \theta\|_{L^1(\Omega)} \leq C_1 \|\tilde{m} - m\|^{\frac{1}{3}}_{L^1(\Omega)}. \tag{2.4}$$

To establish this assertion, by the equation of $\tilde{\theta}$ and θ we have

$$\int_\Omega \left[\mu \nabla(\tilde{\theta} - \theta) \cdot \nabla \psi - (m - 2u)(\tilde{\theta} - \theta)\psi\right] dx + \int_\Omega (\tilde{\theta} - \theta)^2 \psi\, dx = \int_\Omega (\tilde{m} - m)\tilde{\theta}\psi\, dx \tag{2.5}$$

for every $\psi \in W^{1,2}(\Omega)$. Setting $\psi = (\tilde{\theta} - \theta)_+$ in (2.5), we have

$$\int_\Omega \left[\mu |\nabla(\tilde{\theta} - \theta)_+|^2 - (m - 2u)(\tilde{\theta} - \theta)_+^2\right] dx + \int_\Omega (\tilde{\theta} - \theta)_+^3\, dx = \int_\Omega (\tilde{m} - m)\tilde{\theta}(\tilde{\theta} - \theta)_+\, dx. \tag{2.6}$$

By (1.1), we see that the least eigenvalue of the operator $-\mu\Delta - m + \theta$ with zero Neumann boundary condition is zero. Hence, by the comparison principle [36], the operator $-\mu\Delta - m + 2\theta$ with zero Neumann boundary condition is positive. By the variational characterization of eigenvalues we see that the first integral of (2.6) is non-negative. Therefore,

$$\begin{aligned}\int_\Omega (\tilde\theta - \theta)_+^3\, dx &\le \|m\|_{L^\infty} \int_\Omega |\tilde m - m|(\tilde\theta - \theta)_+\, dx \\ &\le \|m\|_{L^\infty} \|\tilde m - m\|_{L^{3/2}} \|(\tilde\theta - \theta)_+\|_{L^3} \\ &\le \|m\|_{L^\infty} \cdot (2\|m\|_{L^\infty})^{\frac{1}{3}} \|\tilde m - m\|_{L^1}^{\frac{2}{3}} \|(\tilde\theta - \theta)_+\|_{L^3},\end{aligned} \tag{2.7}$$

from which it follows that

$$\int_\Omega (\tilde\theta - \theta)_+^3\, dx \le C_2 \int_\Omega |\tilde m - m|\, dx, \tag{2.8}$$

where C_2 is some positive constant independent of ϵ and μ. Applying Hölder inequality again we get

$$\int_\Omega (\tilde\theta - \theta)_+\, dx \le C_3 \left(\int_\Omega |\tilde m - m|\, dx \right)^{\frac{1}{3}}. \tag{2.9}$$

The estimate for $\int_\Omega (\tilde\theta - \theta)_-$ is identical. This completes the proof of (2.4).

By (2.4) we have

$$\begin{aligned}\|\theta - m_+\|_{L^1(\Omega)} &\le \|\theta - \tilde\theta\|_{L^1(\Omega)} + \|\tilde\theta - \tilde m_+\|_{L^1(\Omega)} + \|\tilde m_+ - m_+\|_{L^1(\Omega)} \\ &\le C_1 \|\tilde m - m\|_{L^1(\Omega)}^{\frac{1}{3}} + \|\tilde\theta - \tilde m_+\|_{L^1(\Omega)} + \|\tilde m_+ - m_+\|_{L^1(\Omega)} \\ &\le C_1 \|\tilde m - m\|_{L^1(\Omega)}^{\frac{1}{3}} + \|\tilde\theta - \tilde m_+\|_{L^1(\Omega)} + \|\tilde m - m\|_{L^1(\Omega)},\end{aligned} \tag{2.10}$$

where the last inequality follows from $|\tilde m_+ - m_+| \le |\tilde m - m|$. Since $\tilde\theta \to \tilde m_+$ uniformly as $\mu \to 0$, we can find $\bar\mu = \bar\mu(\epsilon)$ such that if $\mu \le \bar\mu$, $\|\tilde\theta - \tilde m_+\|_{L^1} \le \epsilon$. This along with (2.1) and (2.10) implies that $\|\theta - m_+\|_{L^1} \le C_1 \epsilon^{\frac{1}{3}} + 2\epsilon$. □

The following result is a consequence of elliptic regularity and Sobolev embedding theorem [16]. Since the proof is rather standard, we omit it.

Lemma 2.2. *If assumption* (A) *holds, then as* $\mu \to \infty$, $\theta(x,\mu) \to \frac{1}{|\Omega|}\int_\Omega m(x)\, dx$ *in* $W^{2,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ *for every* $p \ge 1$ *and every* $\alpha \in (0,1)$.

We are now ready for

Proof of Theorem 1.2. Recall that u is a weak solution of (1.1), i.e.,

$$\mu \int_\Omega \nabla\theta \cdot \nabla\varphi \, dx = \int_\Omega \theta[m(x) - \theta]\varphi \, dx \tag{2.11}$$

for any $\varphi \in W^{1,2}(\Omega)$. Since u is positive in $\bar{\Omega}$ (via Harnack inequality [16]) and $u \in C^1(\overline{\Omega})$, we can set $\varphi = u^{-1}$ in (2.11) to get

$$\int_\Omega \theta(x,\mu)\, dx = \int_\Omega m(x)\, dx + \mu \int_\Omega \frac{|\nabla\theta|^2}{\theta^2}\, dx > \int_\Omega m(x)\, dx, \tag{2.12}$$

where the last inequality is strict since $\theta(x,\mu)$ is non-constant: if $\theta(x,\mu)$ were a constant function, then $m(x)$ must also be a constant function, which contradicts assumption (A). This proves part (a). By Lemmas 2.1 and 2.2 we see that if $m(x) \geq 0$, $F(0) = F(\infty) = \int_\Omega m(x)\, dx$. This together with part (a) implies part (b). □

Remark 2.3. We conjecture that if assumption (A) holds and m is non-negative, then there exists a unique $\mu^* > 0$ such that $F(\mu)$ is increasing in $(0,\mu^*)$ and decreasing in (μ^*,∞).

3. The population size of a single species: sign-changing $m(x)$

In this section, we study the case when $m(x)$ changes sign and $n = 1$ and establish Theorem 1.5. We first consider the case when $m(x) = 0$ has exactly one root in $(0,1)$. Without loss of generality, we may assume that there exists $a \in (0,1)$ such that $m(x) < 0$ in $(0,a)$, $m(x) > 0$ in $(a,1)$, and $m'(a) = 1$. We shall return to the general case in the end of this section.

For $x \in [0,1]$, define

$$h_1(x) = m(x) - c_2\mu^{\frac{1}{3}} e^{c_1\mu^{-\frac{1}{3}}(x-\bar{x})} - \mu^{\frac{2}{3}} g(x,\mu), \tag{3.1a}$$

$$h_2(x) = h_1(x) - c_1^2\mu^{\frac{1}{3}}, \tag{3.1b}$$

where $g(x,\mu) \in C^2[a,1]$ such that

$$g(x,\mu) \equiv \begin{cases} c_4, & a \leq x \leq a + 2c_1^2\mu^{\frac{1}{3}}, \\ \in [c_3, c_4], & a + 2c_1^2\mu^{\frac{1}{3}} \leq x \leq a + 3c_1^2\mu^{\frac{1}{3}}, \\ c_3, & a + 3c_1^2\mu^{\frac{1}{3}} \leq x \leq 1, \end{cases} \tag{3.2}$$

and there exists some universal constant K_0 such that

$$\|g_x\|_{L^\infty(a,1)} \le \frac{K_0 c_4}{c_1^2 \mu^{\frac{1}{3}}}, \tag{3.3a}$$

$$\|g_{xx}\|_{L^\infty(a,1)} \le \frac{K_0 c_4}{c_1^4 \mu^{\frac{2}{3}}}, \tag{3.3b}$$

and c_i $(1 \le i \le 4)$ are positive constants which will be determined along the proof in the following manner: c_1 will be chosen sufficiently large; both c_2 and c_3 will be determined by c_1 and are of the order c_1^{-1}; c_4 is also determined by c_1 and is of the order c_1^4.

We choose a point $\bar{x} \in (a, 1]$ as follows: If m is monotone increasing in $(a, 1)$, we define $\bar{x} = 1$; otherwise, $m'(x)$ is negative somewhere in $(a, 1)$. For this case, since $\theta(x) \to m(x)$ uniformly in $[a, 1]$ as $\mu \to 0+$, we can find a critical point of θ in $(a, 1)$, still denoted by $\bar{x}$, such that $\overline{x} \ge a + \delta$ and $m(\overline{x}) \ge \delta$ for sufficiently small μ and some $\delta > 0$ which is independent of μ.

Lemma 3.1. *For sufficiently small μ, $h_2(x) = 0$ has a unique root in $[a, \bar{x}]$, denoted by $x^* = x^*(\mu, c_i)$.*

Proof. Since $m(a) = 0$, we have $h_2(a) < 0$. For sufficiently small μ,

$$h_2(\bar{x}) = m(\bar{x}) - c_1^2 \mu^{\frac{1}{3}} - c_2 \mu^{\frac{1}{3}} - c_4 \mu^{\frac{2}{3}} \ge \tfrac{1}{2} m(\bar{x}) \ge \tfrac{1}{2}\delta > 0.$$

Hence, $h_2(x) = 0$ has at least one root in $(a, \bar{x})$.

To show the uniqueness, we note that

$$h_2'(x) = m'(x) - c_1 c_2 e^{c_1 \mu^{-\frac{1}{3}}(x - \bar{x})} - \mu^{\frac{2}{3}} g_x. \tag{3.4}$$

Hence, $h_2'(x)$ is uniformly bounded for x close to but less than $\bar{x}$, which implies that all roots of h_2 are uniformly bounded away from $\bar{x}$.

We claim that if $\hat{x} = \hat{x}(\mu)$ is a root of h_2, then $\hat{x} \to a$ as $\mu \to 0+$.

To show our assertion, since g is bounded and $\hat{x}$ is uniformly bounded away from $\bar{x}$, we have $m(\hat{x})/\mu^{\frac{1}{3}} \to c_1^2$ as $\mu \to 0$. Since $m'(a) = 1$ and $m(a) = 0$, we thus have $(\hat{x} - a)/\mu^{\frac{1}{3}} \to c_1^2$ as $\mu \to 0$. In particular, $\hat{x} \le a + 2c_1^2 \mu^{\frac{1}{3}}$ for sufficiently small μ. By (3.3a), (3.4), and $m'(a) = 1$ we see that there exists $\delta_1 > 0$ such that $h_2'(x) \ge \frac{1}{2}$

for $x \in [a, a+\delta_1]$ and sufficiently small μ. In particular, this implies that $h_2 = 0$ has a unique root in $[a, a+\delta_1]$, which in turn establishes the uniqueness in the whole interval $[a, \bar{x}]$ for sufficiently small μ. In particular, we have $h_2(x) > 0$ for $x > x^*$. □

Our first primary goal is to prove

Proposition 3.2. *There exists $K_1 > 0$ (independent of μ and c_1) such that for every $c_1 \geq K_1$, there exists $\overline{\mu} = \overline{\mu}(c_1)$ such that if $\mu \leq \overline{\mu}$, we have*

$$\int_{x^*}^{1} [\theta(x) - m(x)]\, dx \geq -\tfrac{2}{c_1}\mu^{\frac{2}{3}}. \tag{3.5}$$

The proof of (3.5) is quite lengthy and is divided into a few steps. Note that u also satisfies the equation

$$\mu\theta_{xx} + \theta[m(x) - \theta] = 0 \qquad \text{in } (0, \bar{x}), \qquad \theta_x(0) = \theta_x(\bar{x}) = 0. \tag{3.6}$$

The main ingredient in the proof of (3.5) is to find a weak subsolution (in H^1 sense) to (3.6), i.e., some function $\underline{\theta} \in H^1(0, \bar{x}) \cap C[0, \bar{x}]$ which satisfies

$$\mu \int_0^{\bar{x}} \underline{\theta}_x \varphi_x\, dx \leq \int_0^{\bar{x}} \underline{\theta}\varphi[m(x) - \underline{\theta}]\, dx, \qquad \forall \varphi \in C^1[0, \bar{x}], \quad \varphi \geq 0. \tag{3.7}$$

Since $m'(a) = 1$ and $c_3 \leq g \leq c_4$, for sufficiently small μ, we have

$$a + c_1^2\mu^{\frac{1}{3}} \leq x^* \leq a + c_1^2\mu^{\frac{1}{3}} + c_5\mu^{\frac{2}{3}} \tag{3.8}$$

for some $c_5 > 0$ which is independent of μ. Hence, $g(x^*, \mu) = c_4$, i.e., x^* satisfies

$$m(x^*) - c_1^2\mu^{\frac{1}{2}} - c_4\mu^{\frac{2}{3}} - c_2\mu^{\frac{1}{3}} e^{c_1\mu^{-\frac{1}{3}}(x^* - \bar{x})} = 0. \tag{3.9}$$

Therefore,

$$x^* = a + c_1^2\mu^{\frac{1}{3}} + \left[c_4 - \tfrac{1}{2}m''(a)c_1^4\right]\mu^{\frac{2}{3}} + o(\mu^{\frac{2}{3}}). \tag{3.10}$$

Let $x_* = x_*(\mu)$ be chosen such that

$$h_1(x^*) + h_1'(x^*)(x_* - x^*) = 0. \tag{3.11}$$

By (3.10) and (3.11) we find

$$x_* = a + \left[c_4 + \tfrac{1}{2}m''(a)c_1^4\right]\mu^{\frac{2}{3}} + o(\mu^{\frac{2}{3}}). \tag{3.12}$$

We construct a weak subsolution (in H^1 sense) to (3.6). In this connection, define

$$\underline{\theta}(x) = \begin{cases} 0, & 0 \le x \le x_*, \\ h_1(x^*) + h_1'(x^*)(x - x^*), & x_* \le x \le x^*, \\ h_1(x), & x^* \le x \le \bar{x}. \end{cases} \tag{3.13}$$

To show that $\underline{\theta}(x)$ is a weak solution, we establish the following properties of $\underline{\theta}$.

Lemma 3.3. *$\underline{\theta} \in H^1(0, \bar{x}) \cap C[0, \bar{x}]$, $\underline{\theta} \ge 0$, and $\underline{\theta}(x) \le m(x)$ for $x \in [x_*, x^*]$.*

Proof. From our construction it is easy to see that $\underline{\theta} \in H^1(0, \bar{x}) \cap C[0, \bar{x}]$. To show that $\underline{\theta} \ge 0$, we see that for $x \in [x_*, x^*]$, $\underline{\theta}(x_*) = 0$ by the definition of x_*; $\underline{\theta}(x^*) = h_1(x^*) = c_1^2 \mu^{\frac{1}{3}} > 0$. Since $\underline{\theta}$ is a linear function in $[x_*, x^*]$, we see that $\underline{\theta} \ge 0$ in $[x_*, x^*]$; if $x \in [x^*, 1]$, $\underline{\theta}(x) = h_1(x) = h_2(x) + c_1^2 \mu^{\frac{1}{3}} \ge c_1^2 \mu^{\frac{1}{3}}$ since $h_2(x) \ge 0$ for $x \ge x^*$.

To verify the last property of $\underline{\theta}$, we set $h_3(x) = m(x) - \underline{\theta}(x)$. Since $g(x, \mu) = c_3$ for $x \in [x_*, x^*]$, we can rewrite h_3 as

$$h_3(x) = h_1(x) + c_2 \mu^{\frac{1}{3}} e^{c_1 \mu^{-\frac{1}{3}}(x - \bar{x})} + c_4 \mu^{\frac{2}{3}} - h_1(x^*) - h_1'(x^*)(x - x^*).$$

Hence, h_3 satisfies

$$h_3(x) \ge c_4 \mu^{\frac{2}{3}} + \left[h_1(x) - h_1(x^*) - h_1'(x^*)(x - x^*)\right]. \tag{3.14}$$

By Taylor's formula, there exists some $\zeta = \zeta(x) \in (x, x^*)$ such that

$$h_3(x) \ge c_4 \mu^{\frac{2}{3}} + \tfrac{1}{2} h_1''(\zeta)(x - x^*)^2. \tag{3.15}$$

Since $g(x; \mu) = c_4$, $\|h_1''(\zeta)\| \le \|m''\|_{L^\infty} + 1$. This together with the estimate $x^* - a \le 2c_1^2 \mu^{\frac{1}{3}}$ gives

$$h_3(x) \ge \left[c_4 - 2c_1^4(\|m''\|_{L^\infty} + 1)\right] \mu^{\frac{2}{3}}, \qquad \forall x \in (x_*, x^*). \tag{3.16}$$

Therefore, $h_3 > 0$ in $[x_*, x^*]$, provided that μ is sufficiently small and

$$c_4 := 3c_1^4(\|m''\|_{L^\infty} + 1). \tag{3.17}$$

This completes the proof of Lemma 3.3. $\square$

Set

$$\begin{aligned} c_1 &\ge \|m''\|_{L^\infty} + 3K_0(\|m''\|_{L^\infty} + 1), \\ c_2 &= \|m'\|_{L^\infty} c_1^{-1}, \\ c_3 &= c_1^{-1}. \end{aligned} \tag{3.18}$$

Lemma 3.4. *If μ is sufficiently small, we have $h_1'(\bar{x}) \le 0$ and*

$$h_1(m-h_1)+\mu h_1'' \ge 0 \qquad in \quad [x^*,\bar{x}]. \tag{3.19}$$

Proof. To check that $h_1'(\bar{x}) \le 0$, we note that for $x \ge a + 3c_1^2\mu^{\frac{1}{3}}$, $g(x,\mu) = c_3$, and thus

$$h_1'(x) = m'(x) - c_1c_2e^{c_1\mu^{-\frac{1}{3}}(x-\bar{x})}. \tag{3.20}$$

Hence, by the choice of c_2, we have $h_1'(\bar{x}) = m'(\bar{x}) - c_1c_2 \le 0$.

To establish (3.19), by direct calculation we have

$$\begin{aligned}
&h_1(m-h_1)+\mu h_1''\\
&= \left(h_2 + c_1^2\mu^{\frac{1}{3}}\right)\left[c_2\mu^{\frac{1}{3}}e^{c_1\mu^{-\frac{1}{3}}(x-\bar{x})} + \mu^{\frac{2}{3}}g\right] + \mu m'' - c_1^2c_2\mu^{\frac{2}{3}}e^{c_1\mu^{-\frac{1}{3}}(x-\bar{x})} - \mu^{\frac{5}{3}}g_{xx}\\
&= c_2\mu^{\frac{1}{3}}e^{c_1\mu^{-\frac{1}{3}}(x-\bar{x})}h_2(x) + \mu^{\frac{2}{3}}g(x)h_2(x) + \mu(c_1^2g + m'' - \mu^{\frac{2}{3}}g_{xx}).
\end{aligned} \tag{3.21}$$

Since $h_2(x) \ge 0$ and $g(x) \ge c_3$ for $x \ge x^*$, by (3.3), (3.18), and (3.21) we have

$$\begin{aligned}
h_1(m-h_1)+\mu h_1'' &\ge \mu\left[c_1^2c_3 - \|m''\|_{L^\infty} - K_0c_4c_1^{-4}\right]\\
&= \mu\left[c_1 - \|m''\|_{L^\infty} - 3K_0(\|m''\|_{L^\infty}+1)\right] \ge 0
\end{aligned} \tag{3.22}$$

by the choices of c_1 and c_3. □

Lemma 3.5. *If μ is sufficiently small, $\underline{\theta}$ is a weak subsolution of* (3.6).

Proof. Since $h_1'(x^*) > 0$, $\varphi \ge 0$, and $h_1'(\bar{x}) \le 0$, we have

$$\begin{aligned}
\mu\int_0^{\bar{x}} \underline{\theta}_x\varphi_x\,dx &= \mu h_1'(x^*)\int_{x_*}^{x^*}\varphi_x\,dx + \mu\int_{x^*}^{\bar{x}} h_1'(x)\varphi_x\,dx\\
&= -\mu h_1'(x^*)\varphi(x_*) + \mu h_1'(\bar{x})\varphi(\bar{x}) - \mu\int_{x^*}^{\bar{x}} h_1''\varphi\,dx\\
&\le -\mu\int_{x^*}^{\bar{x}} h_1''\varphi\,dx.
\end{aligned} \tag{3.23}$$

By Lemma 3.3, $m \ge \underline{\theta}$ in $[x_*, x^*]$. Hence, we have

$$\begin{aligned}
\int_0^{\bar{x}} \underline{\theta}\varphi[m(x)-\underline{\theta}]\,dx &= \int_{x^*}^{\bar{x}} \underline{\theta}\varphi[m(x)-\underline{\theta}]\,dx + \int_{x_*}^{x^*} \underline{\theta}\varphi[m(x)-\underline{\theta}]\,dx\\
&\ge \int_{x^*}^{\bar{x}} h_1\varphi[m(x)-h_1]\,dx.
\end{aligned} \tag{3.24}$$

Therefore, by (3.23), (3.24), and Lemma 3.4 we have

$$\int_0^{\bar{x}} \underline{\theta}\varphi[m(x)-\underline{\theta}]\,dx - \mu\int_0^{\bar{x}} \underline{\theta}_x\varphi_x\,dx \geq \int_{x^*}^{\bar{x}} [h_1(m-h_1)+\mu h_1''(x)]\,\varphi\,dx \geq 0.$$

This proves Lemma 3.5. □

We are now ready for

Proof of Proposition 3.2. Since $\underline{\theta}$ is a weak subsolution of (3.6), by the sub-super solution method [34] we have $\theta(x) \geq \underline{\theta}(x)$ for every $x \in [0,\bar{x}]$. In particular, this implies that

$$\begin{aligned}\int_{x^*}^{\bar{x}} [\theta(x)-m(x)]\,dx &\geq \int_{x^*}^{\bar{x}} [\underline{\theta}(x)-m(x)]\,dx \\ &= -c_2\mu^{\frac{1}{3}}\int_{x^*}^{\bar{x}} e^{c_1\mu^{-\frac{1}{3}}(x-\bar{x})}\,dx - \mu^{\frac{2}{3}}\int_{x^*}^{\bar{x}} g(x,\mu)\,dx \\ &\geq -c_2c_1^{-1}\mu^{\frac{2}{3}} - \mu^{\frac{2}{3}}\int_{x^*}^{\bar{x}} g(x,\mu)\,dx.\end{aligned} \tag{3.25}$$

Since $x^* \in (a+c_1^2\mu^{\frac{1}{3}}, a+3c_1^2\mu^{\frac{1}{3}})$, by (3.2) we have

$$\int_{x^*}^{\bar{x}} g(x,\mu)\,dx \leq \int_{x^*}^{a+3c_1^2\mu^{\frac{1}{3}}} c_4 + \int_{a+3c_1^2\mu^{\frac{1}{3}}}^{\bar{x}} c_3 \leq 2c_1c_4\mu^{\frac{1}{3}} + c_3. \tag{3.26}$$

By (3.17), (3.18), (3.25), and (3.26) we get

$$\begin{aligned}\int_{x^*}^{\bar{x}} [\theta(x)-m(x)]\,dx &\geq -(c_2c_1^{-1}+c_3)\mu^{\frac{2}{3}} - 2c_1c_4\mu \\ &= -(\|m'\|_{L^\infty}c_1^{-2}+c_1^{-1})\mu^{\frac{2}{3}} - 6c_1^5(\|m''\|_{L^\infty}+1)\mu.\end{aligned} \tag{3.27}$$

Since $\theta_x(1) = \theta_x(\bar{x}) = 0$, we have

$$\int_{\bar{x}}^1 [\theta - m(x)]\,dx = \mu\int_{\bar{x}}^1 \frac{\theta_{xx}}{\theta}\,dx = \mu\int_{\bar{x}}^1 \frac{\theta_x^2}{\theta^2}\,dx > 0. \tag{3.28}$$

By choosing

$$c_1 \geq 2\|m'\|_{L^\infty}, \tag{3.29a}$$

$$\mu \leq \left[12c_1^6(\|m''\|_{L^\infty}+1)\right]^{-3}, \tag{3.29b}$$

we see that (3.5) follows from (3.27), (3.28), and (3.29). This completes the proof of Proposition 3.2. □

It remains to estimate $\int_a^{x^*}[\theta(x)-m(x)]$ and in this connection we have

Proposition 3.6. *There exists $K_2 > 0$ (independent of μ, c_1) such that for every $c_1 \geq K_2$, there exists $\tilde{\mu} = \tilde{\mu}(c_1)$ such that if $\mu \leq \tilde{\mu}$, we have*

$$\int_a^{x^*} [\theta(x) - m(x)]\, dx \geq g_1(c_1)\mu^{\frac{2}{3}}, \tag{3.30}$$

where g_1 is some strictly increasing function with $g_1(0) = 0$.

Note that $x^* - a$ is of the order $\mu^{\frac{1}{3}}$. Thus it is natural to introduce the new variable $y = \mu^{-\frac{1}{3}}(x - a)$ and set $v(y, \mu) = \mu^{-\frac{1}{3}}\theta(\mu^{\frac{1}{3}}y + a)$ for $y \in (-a\mu^{-\frac{1}{3}},\ \ (1-a)\mu^{-\frac{1}{3}})$. Then v satisfies

$$v_{yy} + v\left[\mu^{-\frac{1}{3}}m(a + \mu^{\frac{1}{3}}y) - v\right] = 0, \qquad y \in (-a\mu^{-\frac{1}{3}},\ \ (1-a)\mu^{-\frac{1}{3}}). \tag{3.31}$$

Lemma 3.7. *There exists some positive constant $c_6 > 0$ such that for $\mu \ll 1$,*

$$-c_6 \leq v(y) - \mu^{-\frac{1}{3}}m_+(a + \mu^{\frac{1}{3}}y) \leq c_6, \qquad y \in (-a\mu^{-\frac{1}{3}},\ \ (1-a)\mu^{-\frac{1}{3}}). \tag{3.32}$$

The proof of (3.32) is almost identical to that of Proposition A.1 in [24]. More precisely, modifying the proof of Proposition A.1 in [24] we have

$$-c_7\mu^{\frac{1}{3}} \leq \theta(x;\mu) - m_+(x) \leq c_7\mu^{\frac{1}{3}}, \tag{3.33}$$

which implies (3.32) after the scaling $x = a + \mu^{\frac{1}{3}}y$. We refer to the proof of Proposition A.1 for the details.

In particular, for every $R > 0$, since $\|\mu^{-\frac{1}{3}}m_+(a + \mu^{\frac{1}{3}}\cdot)\|_{L^\infty(-R,R)}$ is uniformly bounded for sufficiently small μ, by (3.32) we see that $\|v(\cdot;\mu)\|_{L^\infty(-R,R)}$ is uniformly bounded for $\mu \ll 1$. Hence, we can apply the smoothness of m and the equation of v to show that for every $R > 0$, $\|v(\cdot,\mu)\|_{C^3[-R,R]}$ is uniformly bounded for $\mu \ll 1$. Hence, by Ascoli-Arzela Lemma and standard diagonal process argument, passing to some subsequence if necessary, we may assume that $v(y,\mu) \to w(y)$ in $C^2([-R,R])$ for every $R > 0$ as $\mu \to 0$, where $w(y)$ is a smooth solution of

$$w_{yy} + w(y_+ - w) = 0, \quad w(y) > 0, \quad |w(y) - y_+| \leq c_6, \qquad y \in (-\infty, \infty). \tag{3.34}$$

Indeed, for every $R > 0$, the convergence of v to w in $C^2([-R,R])$ is independent of the sequence of μ due to the following result.

Lemma 3.8. (3.34) *has a unique solution and it also satisfies* $w(y) > y_+$.

Proof. We first show that every solution of (3.34) satisfies $w(y) > y_+$. To this end, we argue by contradiction: Suppose that there exists $y_0 > 0$ such that $w(y) > y_+$ for $y < y_0$ and $w(y_0) = y_0$. Hence, $w'(y_0) \leq 1$. If $w'(y_0) = 1$, both functions $w(y)$ and y satisfy

$$z_{yy} + z(y - z) = 0, \qquad z(y_0) = y_0, \qquad z_y(y_0) = 1. \tag{3.35}$$

Therefore, $w(y) \equiv y$, which is impossible. This implies that $w_y(y_0) < 1$. Hence, for $y > y_0$ but close to y_0, we have $w(y) < y$. There are two possibilities here:
(a) There exists another $y_1 > y_0$ such that $w(y) < y$ for $y \in (y_0, y_1)$ and $w(y_1) = y_1$. We show that this case can not occur. To this end, set $z(y) = y - w(y)$. Then $z(y)$ satisfies

$$z_{yy} = [y - z(y)]z(y) > 0, \quad z(y) > 0 \quad \text{in } (y_0, y_1), \qquad z(y_0) = z(y_1) = 0. \tag{3.36}$$

It is obvious that such z does not exist.
(b) $w(y) < y$ for every $y > y_0$. Since $w(y) < y$, we see that $w_{yy} < 0$ for $y > y_0$. Therefore, $w_y(y) \leq w_y(y_0)$ for $y > y_0$. In particular, this ensures that

$$w(y) \leq w_y(y_0)(y - y_0) + w(y_0) \tag{3.37}$$

for $y \geq y_0$. Since $w(y) \geq y - c_6$ for some $c_6 > 0$, we have

$$y - c_6 \leq w_y(y_0)(y - y_0) + w(y_0) \tag{3.38}$$

for $y \geq y_0$, which is a contradiction since $w_y(y_0) < 1$. This completes the proof of $w(y) > y_+$.

It remains to prove the uniqueness of w. Again, we argue by contradiction: Suppose that (3.34) has two solutions, denoted by w and v such that $w \not\equiv v$. We first show that $w - v$ can not change sign more than once. If not, we may assume that there exist $a < b$ such that $w - v > 0$ in (a, b), $w(a) = v(a)$ and $w(b) = v(b)$. It is easy to check that

$$[vw_y - wv_y]_y + wv(v - w) = 0 \qquad y \in (-\infty, +\infty). \tag{3.39}$$

Note that $w'(a) \geq v'(a)$ and $w'(b) \leq v'(b)$. Integrating (3.39) in (a, b) and using $w(a) = v(a)$ and $w(b) = v(b)$ we get

$$w(b)[w'(b) - v'(b)] - w(a)[w'(a) - v'(a)] + \int_a^b wv[v - w]\, dx = 0, \tag{3.40}$$

which is a contradiction. Hence, $w - v$ does not change sign more than once.

We further show that $w - v$ does not change sign. If not, we may assume that there exists b such $w > v$ in $(-\infty, b)$ and $w(b) = v(b)$. It is easy to see that for any solution w of (3.3) we have $w(y) \to 0$ and $w'(y) \to 0$ as $y \to -\infty$. By letting $a \to -\infty$ in (3.40) and applying $w'(b) \leq v'(b)$, we again reach a contradiction.

Therefore, without loss of generality, we may assume that $w > v$ in R^1. By (3.39) we see that

$$[vw_y - wv_y]_y > 0, \qquad y \in (-\infty, \infty). \tag{3.41}$$

Since $vw_y - wv_y \to 0$ as $y \to -\infty$, we see that $vw_y - wv_y > 0$ in R^1, which implies that $w(y)/v(y)$ is strictly increasing. By (3.34), $w(y)/y \to 1$ and $v(y)/y \to 1$ as $y \to +\infty$. Hence, $w(y)/v(y) \to 1$ as $y \to \infty$. Therefore, $w(y)/v(y) < 1$ in R^1. However, this contradicts our assumption $w > v$. Hence, (3.34) has a unique solution. □

We are now ready for

Proof of Proposition 3.6. By (3.8) we see that $(x^* - a)\mu^{-\frac{1}{3}} \to c_1^2$ as $\mu \to 0+$. Since $v(y, \mu) \to w(y)$ uniformly in any compact subset of R^1 and $v(y, \mu) - \mu^{-\frac{1}{3}} m(a + \mu^{\frac{1}{3}} y)$ is bounded and converges to $w(y) - y_+$ pointwisely as $\mu \to 0$, by Lebesgue dominant convergence theorem we have

$$\begin{aligned} \mu^{-\frac{2}{3}} \int_a^{x^*} [\theta(x) - m(x)]\, dx &= \int_0^{(x^*-a)\mu^{-\frac{1}{3}}} \left[v(y; \mu) - \mu^{-\frac{1}{3}} m(a + \mu^{\frac{1}{3}} y)\right] dy \\ &\to \int_0^{c_1^2} [w(y) - y]\, dy \end{aligned} \tag{3.42}$$

as $\mu \to 0+$. In particular, for sufficiently small μ, we have

$$\int_a^{x^*} [\theta(x) - (x - a)]\, dx \geq g_1(c_1)\mu^{\frac{2}{3}}, \tag{3.43}$$

where

$$g_1(c_1) \equiv \tfrac{1}{2} \int_0^{c_1^2} [w(y) - y]\, dy. \tag{3.44}$$

Since $w(y) > y_+$, $g_1(c_1)$ is strictly increasing in c_1. This completes the proof of Proposition 3.6. $\square$

We are now ready to complete

Proof of Theorem 1.5. For the case when $m(x) = 0$ changes sign exactly once in $(0,1)$, by Propositions 3.2 and 3.6 we have

$$\begin{aligned} F(\mu) - F(0) &= \int_0^1 [\theta(x) - m_+(x)]\, dx \\ &\geq \int_a^{x^*} [\theta(x) - m(x)]\, dx + \int_{x^*}^1 [\theta(x) - m(x)]\, dx \\ &\geq [g(c_1) - \tfrac{2}{c_1}]\mu^{\frac{2}{3}} \\ &\geq \tfrac{1}{c_1}\mu^{\frac{2}{3}}, \end{aligned} \tag{3.45}$$

where c_1 is chosen large such that

$$c_1 g(c_1) \geq 3. \tag{3.46}$$

For the general case, by the assumption we may write the roots of $m(x) = 0$ in $(0,1)$ as follows: $0 < x_1 < ... < x_k < 1$, $k \geq 2$. If $\mu \ll 1$, $\theta(x;\mu)$ is sufficiently close to m_+ in $C[0,1]$. Hence, for each interval (x_i, x_{i+1}), $1 \leq i \leq k-1$, there exists $x_i^* = x_i^*(\mu) \in (x_i, x_{i+1})$ such that $\theta_x(x_i^*) = 0$. Moreover, there exists $\delta_i > 0$ such that for sufficiently small μ, if $m(x_i^*) > 0$, we can choose x_i^* such that $x_i + \delta_i \leq x_i^* \leq x_{i+1} - \delta_i$ and $m(x_i^*) \geq \delta_i$. This is possible since $m'(x_i) \neq 0$, $m'(x_{i+1}) \neq 0$, and $\theta(x) \to m_+(x)$ uniformly as $\mu \to 0$.

Set $x_0^* = 0$ and $x_{k+1}^* = 1$. Note that for every $0 \leq i \leq k$, θ satisfies

$$\mu\theta_{xx} + \theta[m(x) - \theta] = 0 \qquad x \in (x_i^*, x_{i+1}^*), \qquad \theta_x(x_i^*) = \theta_x(x_{i+1}^*) = 0. \tag{3.47}$$

Note that $m(x)$ changes sign exactly once in (x_i^*, x_{i+1}^*), namely, at $x = x_i$. Then, for every subinterval (x_i^*, x_{i+1}^*), repeating the exact same proof as in the case when m has exactly one root in (0, 1), we find $c_{0,i} > 0$ and $\mu_i > 0$ such that

$$\int_{x_i^*}^{x_{i+1}^*} [\theta(x) - m_+(x)]\, dx \geq c_{0,i}\mu^{\frac{2}{3}}, \qquad \forall \mu < \mu_i. \tag{3.48}$$

Set $\mu_0 = \min_{0 \leq i \leq k} \mu_i$ and $c_0 = \sum_{i=0}^k c_{0,i}$, we see that $F(\mu) \geq F(0) + c_0\mu^{\frac{2}{3}}$ for $\mu < \mu_0$. The proof of Theorem 1.5 is now complete. $\square$

4. The linear stability of $(\theta, 0)$

In this section, we study the stability of the semi-trivial steady state $(\theta, 0)$ of (1.5) and the main goal is to establish Theorems 1.7 and 1.10. We first present

Proof of Theorem 1.7. For every $\mu > 0$, define

$$G(\mu) = \frac{\int_\Omega m(x)\,dx}{\int_\Omega \theta(x,\mu)\,dx}, \tag{4.1}$$

and set

$$c_* = \inf_{0<\mu<\infty} G(\mu). \tag{4.2}$$

By Theorem 1.2, we have

$$G(0) = G(\infty) = 1, \qquad G(\mu) < 1 \text{ for every } \mu \in (0,\infty). \tag{4.3}$$

Hence, $c_* \in (0,1)$.

The stability of $(\theta, 0)$ is determined by the sign of the smallest eigenvalue λ_1 of the linear problem

$$\nu\Delta\varphi + [m(x) - c\theta]\varphi = -\lambda\varphi \qquad \text{in} \quad \Omega, \qquad \frac{\partial\varphi}{\partial n}\Big|_{\partial\Omega} = 0. \tag{4.4}$$

Let φ_1 denote the eigenfunction corresponding to λ_1 with $\sup_\Omega \varphi_1 = 1$, i.e.,

$$\nu\Delta\varphi_1 + [m(x) - c\theta]\varphi_1 = -\lambda_1\varphi_1 \qquad \text{in} \quad \Omega, \tag{4.5a}$$

$$\varphi_1 > 0 \qquad \text{in } \Omega, \qquad \frac{\partial\varphi_1}{\partial n}\Big|_{\partial\Omega} = 0. \tag{4.5b}$$

We first prove part (a). If $c < c_*$, we have $c\int_\Omega \theta(x,\mu)\,dx < \int_\Omega m(x)\,dx$ for every $\mu \in (0,\infty)$. Dividing (4.5a) by φ_1 and integrating in Ω, since φ_1 is not a constant function, we get

$$\lambda_1|\Omega| = -\nu\int_\Omega \frac{|\nabla\varphi_1|^2}{\varphi_1^2}\,dx + \int_\Omega [c\theta(x,\mu) - m]\,dx < \int_\Omega [c\theta(x,\mu) - m]\,dx < 0 \tag{4.6}$$

for every $\mu > 0$ and $\nu > 0$. This proves part (a).

Next we establish part (b). By (4.2) and (4.3) we see that if $c \in (c_*, 1)$, $G(\mu) = c$ has at least two and at most finite many roots, denoted by $\mu_1 \le \mu_2 \le \ldots \le \mu_{2k-1} \le$

μ_{2k} (counting multiplicity) with $k \geq 1$, and $G(\mu) < c$ for $\mu \in \cup_{i=1}^{k}(\mu_{2i-1}, \mu_{2i})$, $G(\mu) > c$ for $\mu \notin \cup_{i=1}^{k}[\mu_{2i-1}, \mu_{2i}]$.

Claim. For every $\mu \in (\mu_{2i-1}, \mu_{2i})$, there exists $\nu_i^* = \nu_i^*(\mu)$ such that $\lambda_1 > 0$ when $\nu > \nu_i^*$, $\lambda_1 = 0$ when $\nu = \nu_i^*$, and $\lambda_1 < 0$ when $\nu < \nu_i^*$. Moreover, for every $1 \leq i \leq k$,

$$\lim_{\mu \to \mu_{2i-1}+} \nu_i^* = \lim_{\mu \to \mu_{2i}-} \nu_i^* = +\infty. \tag{4.7}$$

To prove our assertion, for every $\mu > 0$, we set

$$\lambda_i^*(\mu) := \inf_{\{\psi \in H^1 : \int_\Omega (m - c\theta)\psi^2\, dx > 0\}} \frac{\int_\Omega |\nabla \psi|^2\, dx}{\int_\Omega (m - c\theta)\psi^2\, dx}. \tag{4.8}$$

If $\mu \in (\mu_{2i-1}, \mu_{2i})$, then $G(\mu) < c$, i.e., $\int_\Omega [m - c\theta]\, dx < 0$. Moreover, since $m - \theta$ is positive somewhere in Ω and $0 < c < 1$, $m - c\theta$ is also positive somewhere in Ω. Therefore, for every $\mu \in (\mu_{2i-1}, \mu_{2i})$, we have $\lambda_i^*(\mu) > 0$ (see, e.g., [5]).

Set $\nu_i^*(\mu) = 1/\lambda_i^*(\mu)$. By (4.8), there exists $\varphi^* > 0$ such that

$$\nu_i^* \Delta \varphi^* + [m - c\theta]\varphi^* = 0 \qquad \text{in } \Omega, \qquad \frac{\partial \varphi^*}{\partial n}\Big|_{\partial\Omega} = 0. \tag{4.9}$$

By (4.5) and (4.9), we see that $\lambda_1 = 0$ when $\nu = \nu_i^*$. Since λ_1 is strictly increasing in ν, we have $\lambda_1 > 0$ when $\nu > \nu_i^*$, and $\lambda_1 < 0$ when $\nu < \nu_i^*$.

To prove (4.7), we argue by contradiction. Suppose not, passing to a subsequence if necessary, we may assume that $\nu_i^*(\mu) \to \nu^* \geq 0$ as $\mu \to \mu_{2i-1}+$. By (4.8) we see that $\lambda_i^*(\mu)$ is uniformly bounded from the above in (μ_{2i-1}, μ_{2i}), i.e., $\nu_i^*(\mu)$ is uniformly bounded from the below by some positive constant in (μ_{2i-1}, μ_{2i}). Hence $\nu^* > 0$. By standard elliptic regularity and Sobolev embedding theorem [16], we have $\varphi^* \to \varphi$ in $C^2(\bar{\Omega})$ as $\mu \to \mu_{2i-1}+$, and $\varphi > 0$ satisfies

$$-\nu^* \Delta \varphi = [m(x) - c\theta(x, \mu_{2i-1})]\, \varphi \qquad \text{in } \Omega, \qquad \frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{4.10}$$

Since $G(\mu_{2i-1}) = 0$, we have $\int_\Omega [m - c\theta(x, \mu_{2i-1})]\, dx = 0$. As $m(x) - c\theta(x, \mu_{2i-1})$ is not identically equal to zero, we see that such φ does not exist, which is a contradiction. This proves (4.7).

Define

$$\overline{\nu}(c) = \min_{1 \leq i \leq k} \inf_{\mu \in (\mu_{2i-1}, \mu_{2i})} \nu_i^*(\mu). \tag{4.11}$$

By our assertion, we have $\overline{\nu}(c) > 0$. To complete the proof of Theorem 1.7, we consider two cases.

Case I. $\nu < \overline{\nu}$. For this case, we have $\nu < \nu_i^*(\mu)$ for every i and every $\mu \in (\mu_{2i-1}, \mu_{2i})$. Hence, $\lambda_1 < 0$ for $\mu \in \cup_{1\le i\le k}(\mu_{2i-1}, \mu_{2i})$ and $\nu < \bar{\nu}$. For $\mu \notin \cup_{1\le i\le k}(\mu_{2i-1}, \mu_{2i})$, we have $G(\mu) \ge c$, and similar to the proof of part (a) we can show that $\lambda_1 < 0$ for any $\nu > 0$. Therefore, if $\nu < \overline{\nu}$, $\lambda_1 < 0$ for any $\mu > 0$. This proves the first part of part (b).

Case II. $\nu > \overline{\nu}$. For this case, there exists some i such that $\nu > \min_{\mu\in(\mu_{2i-1},\mu_{2i})} \nu_i^*(\mu)$. We claim that $\nu^*(\mu) - \nu$ changes sign at least twice in (μ_{2i-1}, μ_{2i}). If this assertion holds, we see that λ_1 changes sign at least twice in (μ_{2i-1}, μ_{2i}), which implies that $(\theta, 0)$ changes stability at least twice as μ varies from 0 to ν. This would establish the rest of part (b).

To prove our previous assertion, since $\nu_i^*(\mu) \to +\infty$ as $\mu \to \mu_{2i-1}$ and $\mu \to \mu_{2i}$, it suffices to exclude the possibility that $\nu_i^*(\mu) \equiv \nu$ in some interval $[\underline{\mu}, \bar{\mu}] \subset (\mu_{2i-1}, \mu_{2i})$: If this were the case, we see that $\lambda_1 \equiv 0$ for $\mu \in [\underline{\mu}, \bar{\mu}]$. Since λ_1 is an analytic function of μ, $\lambda_1 \equiv 0$ for $\mu \in (\mu_{2i-1}, \mu_{2i})$, which implies that $\nu_i^*(\mu) \equiv \nu$ in (μ_{2i-1}, μ_{2i}). However, this contradicts (4.7). $\square$

The rest of this section is devoted to

Proof of Theorem 1.10. The proof of Theorem 1.10 is quite similar to that of Theorem 1.7. Define $G(\mu)$ and c_* as in (4.1) and (4.2), respectively. By Theorem 1.5, we have

$$G(0) = \frac{\int_\Omega m\,dx}{\int_\Omega m_+\,dx}, \qquad G(\infty) = 1, \qquad \inf_{0<\mu<\infty} G(\mu) < G(0), \tag{4.12}$$

and $G(\mu) < 1$ for every $\mu \in (0, \infty)$. Hence,

$$c_* \in \left(0, \frac{\int_\Omega m\,dx}{\int_\Omega m_+\,dx}\right) \subset (0, 1). \tag{4.13}$$

The proof of part (a) is exactly the same as that of part (a) of Theorem 1.7 and is omitted.

For part (b), by (4.13) we see that if $c \in (c_*, \int_\Omega m\,dx/\int_\Omega m_+\,dx)$, $G(\mu) = c$ has at least two and at most finite many roots. The rest of proof is exactly the same as that of part (b) of Theorem 1.7 and is also omitted.

For part (c), we see that if $c \in (\int_\Omega m\,dx/\int_\Omega m_+\,dx, 1)$, $G(\mu) = c$ has at least one and at most finite many roots, denoted by $\mu_1 \le \mu_2 \le \dots \le \mu_{2k-1}$ (counting multiplicity). Set $\mu_0 = 0$. We see that $G(\mu) < c$ for $\mu \in \cup_{i=1}^k (\mu_{2i-2}, \mu_{2i-1})$, and $G(\mu) > c$ for $\mu \notin \cup_{i=1}^k [\mu_{2i-2}, \mu_{2i-1}]$.

For every $\mu \in (\mu_{2i-2}, \mu_{2i-1})$ and $1 \le i \le k$, define $\lambda_i^*(\mu)$ as in (4.8). Similarly as in the proof of Theorem 1.7, we see that for every $\mu \in (\mu_{2i-2}, \mu_{2i-1})$, we have $\lambda_i^*(\mu) > 0$. Set $\nu_i^*(\mu) = 1/\lambda_i^*(\mu)$. Again, we have $\lambda_1 > 0$ when $\nu > \nu_i^*$, and $\lambda_1 < 0$ when $\nu < \nu_i^*$.

Define

$$\overline{\nu}(c) = \min_{1\le i\le k} \inf_{\mu\in(\mu_{2i-2},\mu_{2i-1})} \nu_i^*(\mu). \tag{4.14}$$

The intervals (μ_{2i-2}, μ_{2i-1}) for $i \ge 2$ can be treated exactly the same as in the proof of Theorem 1.7; for the interval $(0, \mu_1)$, $\lim_{\mu\to\mu_1-} \nu_i^*(\mu) = +\infty$ still holds; however, $\lim_{\mu\to 0+} \nu_i(\mu)$ is finite. To determine its value, set

$$\hat{\lambda} = \inf_{\{\psi\in H^1 : \int_\Omega (m-cm_+)\psi^2\,dx>0\}} \frac{\int_\Omega |\nabla\psi|^2\,dx}{\int_\Omega (m-cm_+)\psi^2\,dx}. \tag{4.15}$$

Since $c \in (\int_\Omega m\,dx/\int_\Omega m_+\,dx, 1)$, we see that $\int_\Omega (m - cm_+)\,dx < 0$. As $m - cm_+$ is positive somewhere, $\hat{\lambda} > 0$. Set $\hat{\nu} = 1/\hat{\lambda}$. Since $\theta \to m_+$ in L^∞ as $\mu \to 0$, we see that if $\mu \to 0$, $\nu_i^*(\mu) \to \hat{\nu} > 0$. In conclusion, we have $\overline{\nu}(c) > 0$.

Similar to the proof of Theorem 1.7, we consider two cases.

Case I. $\nu < \overline{\nu}$. This case is the same as that in the proof of Theorem 1.7.

Case II. $\nu > \overline{\nu}$. For this case, there exists some i such that $\nu > \min_{\mu\in(\mu_{2i-2},\mu_{2i-1})} \nu_i^*(\mu)$. It is easy to see that $\nu^*(\mu) - \nu$ changes sign at least once in (μ_{2i-1}, μ_{2i}), and thus $(\theta, 0)$ changes stability at least once as μ varies from 0 to ν. This proves part (c) of Theorem 1.10, and the proof of Theorem 1.10 is complete. □

5. Global asymptotic stability of $(\theta, 0)$

It is proved in Section 4 that for every $c \in (c_*, 1)$, the set Σ_c is nonempty. In this subsection, we study the global asymptotic stability of $(\theta, 0)$ for every $(\mu, \nu) \in \Sigma_c$. Before proving Theorem 1.9, we first establish

Lemma 5.1. *If assumption* (A) *holds and* $m(x)$ *is non-negative, then for every* $c \in (c_*, 1)$*, there exists a small positive constant* $\delta = \delta(c, \Omega, m)$ *such that if* $b \in (0, \delta)$ *and* $(\mu, \nu) \in \Sigma_c$*, then* $(\theta, 0)$ *is globally asymptotically stable.*

Proof. By Theorem 1.7, we see that for $c \in (c_*, 1)$ and $(\mu, \nu) \in \Sigma_c$, $(\theta, 0)$ is stable. From the proof of Theorem 1.7, for every $(\mu, \nu) \in \Sigma_c$, we have $\mu < \nu$. Hence, if $b \in (0, 1)$, the other semi-trivial steady state $(0, \theta(x, \nu))$ of (1.5) is unstable. Therefore, since (1.5) is a monotone system, it suffices to show that for every $c \in (c_*, 1)$, there exists $\delta > 0$ such that if $b \in (0, \delta)$ and $(\mu, \nu) \in \Sigma_c$, (1.5) has no positive steady state ([19, 21]). To this end, we argue by contradiction. Suppose not, we may assume that there exist sequences b_i, μ_i, and ν_i such that $b_i \to 0+$, $(\mu_i, \nu_i) \in \Sigma_c$, and (1.5) with $(b, \mu, \nu) = (b_i, \mu_i, \nu_i)$ has a positive steady state (u_i, v_i). Set $\theta_i = \theta(\cdot, \mu_i)$. Since $(\mu_i, \nu_i) \in \Sigma_c$, the principal eigenvalue λ_i of the linear problem

$$-\nu_i \Delta \varphi + [-m(x) + c\theta_i]\varphi = \lambda \varphi \qquad \text{in } \Omega, \qquad \frac{\partial \varphi}{\partial n}\Big|_{\partial\Omega} = 0 \tag{5.1}$$

satisfies $\lambda_i > 0$. Moreover, we can choose the corresponding eigenfunction φ_i such that $\varphi_i > 0$, $\sup_\Omega \varphi_i = 1$, and φ_i satisfies

$$-\nu_i \Delta \varphi_i + [-m(x) + c\theta_i]\varphi_i = \lambda_i \varphi_i \qquad \text{in } \Omega, \qquad \frac{\partial \varphi_i}{\partial n}\Big|_{\partial\Omega} = 0 \tag{5.2}$$

Dividing (5.2) by φ_i and integrating in Ω, we have

$$\int_\Omega (c\theta_i - m)\, dx = \nu_i \int_\Omega \frac{|\nabla \varphi_i|^2}{\varphi_i^2}\, dx + \lambda_i |\Omega| > 0. \tag{5.3}$$

Hence, $c \int_\Omega \theta_i \, dx \geq \int_\Omega m \, dx$. Since $c \in (c_*, 1)$, by part (b) of Theorem 1.1, (4.1), and (4.2), we see that there exist positive constants $\underline{\mu}$ and $\overline{\mu}$ such that $\mu_i \in [\underline{\mu}, \overline{\mu}]$ for every i. Passing to a subsequence if necessary, we may assume that $\mu_i \to \hat{\mu}$ for some $\hat{\mu} > 0$ as $i \to \infty$.

Set $\hat{\theta} = \theta(\cdot, \hat{\mu})$. Since $c\theta_i - m \to c\hat{\theta} - m$ in L^∞, and $c\hat{\theta} - m$ is negative somewhere in Ω, if $\nu_i \to 0$, by the proof of Lemma 3.1 in [22], we have $\lambda_i \to \min_{\overline{\Omega}}[c\hat{\theta} - m] < 0$. However, this is a contradiction since $\lambda_i > 0$. Therefore, there exists $\underline{\nu} > 0$ such that $\nu_i \geq \underline{\nu}$ for sufficiently large i. Again, passing to a subsequence if necessary, we may assume that $\nu_i \to \hat{\nu}$ for some $\hat{\nu} \in (0, \infty]$ as $i \to \infty$.

By the maximum principle, both u_i and v_i are uniformly bounded. Hence, by standard elliptic regularity and Sobolev embedding theorem, passing to some subsequence if necessary, we may assume that $(u_i, v_i) \to (\hat{u}, \hat{v})$ in L^∞.

Set $\tilde{v}_i = v_i/\|v_i\|_\infty$. Dividing the equation of v_i by $\|v_i\|_\infty$, again by elliptic regularity and Sobolev embedding theorem, we may assume that $\tilde{v}_i \to \tilde{v}$ in L^∞. Hence, $\tilde{v}$ satisfies $\tilde{v} \geq 0$, $\sup_\Omega \tilde{v} = 1$, and

$$\hat{\nu}\Delta\tilde{v} + \tilde{v}(m - c\hat{u} - \hat{v}) = 0 \qquad \text{in } \Omega, \qquad \frac{\partial \tilde{v}}{\partial n}\Big|_{\partial\Omega} = 0. \tag{5.4}$$

By Harnack inequality ([16]), we see that $\tilde{v} > 0$ in Ω.

Claim. $\|u_i - \theta_i\|_2/\|v_i\|_\infty \to 0$ as $i \to \infty$.

By the equation of u_i and θ_i, we see that $u_i - \theta_i$ satisfies

$$\mu_i\Delta(u_i - \theta_i) + [m(x) - u_i - \theta_i](u_i - \theta_i) = b_i u_i v_i \qquad \text{in } \Omega. \tag{5.5}$$

Multiplying (5.5) by $u_i - \theta_i$ and integrating in Ω, we get

$$\int_\Omega [\mu_i|\nabla(u_i - \theta_i)|^2 + (-m + u_i + \theta_i)(u_i - \theta_i)^2]\,dx = -b_i \int_\Omega u_i v_i (u_i - \theta_i)\,dx. \tag{5.6}$$

Since $\mu_i \to \hat{\mu}$, $u_i \to \hat{\theta}$, and $\theta_i \to \hat{\theta}$, and the smallest eigenvalue of the operator $-\hat{\mu}\Delta - m + 2\hat{\theta}$ is strictly positive, we see that the smallest eigenvalue of the operator $-\mu_i\Delta - m + u_i + \theta_i$ is uniformly bounded from below by some positive constant γ. Hence, it follows from (5.6) that

$$\gamma \int_\Omega (u_i - \theta_i)^2\,dx \leq b_i \int_\Omega u_i v_i |u_i - \theta_i|\,dx \leq b_i \|u_i\|_\infty \|v_i\|_\infty \int_\Omega |u_i - \theta_i|\,dx. \tag{5.7}$$

Hence, by Cauchy-Schwartz inequality, we get

$$\|u_i - \theta_i\|_2 \leq |\Omega|^{1/2} b_i \|u_i\|_\infty \|v_i\|_\infty/\gamma. \tag{5.8}$$

Since $b_i \to 0$ and u_i is uniformly bounded, we have $\|u_i - \theta_i\|_2/\|v_i\| \to 0$ as $i \to \infty$. This proves our assertion.

Multiplying the equation of v_i by φ_i, integrating in Ω, by (5.2) and $\lambda_i > 0$ we get

$$c\int_\Omega v_i\varphi_i(\theta_i - u_i)\,dx \geq \int_\Omega v_i^2\varphi_i\,dx. \tag{5.9}$$

Dividing both sides of (5.9) by $\|v_i\|_\infty^2$, we have

$$c\int_\Omega \tilde{v}_i\varphi_i\frac{\theta_i-u_i}{\|v_i\|_\infty}\,dx \geq \int_\Omega \tilde{v}_i^2\varphi_i\,dx. \tag{5.10}$$

By elliptic regularity, we see that $\varphi_i \to \varphi$ in L^∞, and $\varphi \geq 0$ and $sup_\Omega\varphi = 1$. Now, passing to the limit in (5.10), by previous assertion we get

$$\int_\Omega \tilde{v}^2\varphi\,dx \leq 0, \tag{5.11}$$

which is a contradiction. This completes the proof of Lemma 5.1. □

We are ready for

Proof of Theorem 1.9. For every $c \in (c_*, 1)$, set

$$b_* = \sup\left\{b \in (0,1] : (1.5) \text{ has no positive steady state for every} (\mu,\nu) \in \Sigma_c\right\}. \tag{5.12}$$

By Lemma 5.1, b_* is well defined and $b_* \in (0,1]$.

Claim. If $c \in (c_*, 1)$, (1.5) has no positive solution for every $b \in (0, b_*)$ and every $(\mu,\nu) \in \Sigma_c$.

To establish our assertion, we argue by contradiction. Suppose that for some $c \in (c_*, 1)$, there exist $\hat{b} \in (0, b_*)$ and $(\hat{\mu},\hat{\nu}) \in \Sigma_c$ such that (1.5) with $(b,\mu,\nu) = (\hat{b},\hat{\mu},\hat{\nu})$ has at least a positive steady state $(\hat{u},\hat{v})$. Then for every $b > \hat{b}$, $(\hat{u},\hat{v})$ satisfies

$$\begin{aligned} &\hat{\mu}\Delta\hat{u} + \hat{u}[m(x) - \hat{u} - b\hat{v}] \leq 0 \qquad \text{in } \Omega,\\ &\hat{\nu}\Delta\hat{v} + \hat{v}[m(x) - \hat{v} - c\hat{u}] = 0 \qquad \text{in } \Omega. \end{aligned} \tag{5.13}$$

Since $(\hat{\mu},\hat{\nu}) \in \Sigma_c$, we have $\hat{\mu} < \hat{\nu}$. By $b < 1$ and the comparision principle we see that $(0, \theta(x,\hat{\nu}))$ is linearly unstable, i.e., the least eigenvalue λ_1 of the linear eigenvalue problem

$$-\hat{\mu}\Delta\varphi + \varphi[-m(x) + b\theta(x,\hat{\nu})] = \lambda_1\varphi \qquad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial n}|_{\partial\Omega} = 0 \tag{5.14}$$

satisfies $\lambda_1 < 0$. Denote the eigenfunction corresponding to λ_1 by φ_1, which can be chosen positive and $\sup_\Omega \varphi_1 = 1$.

Set $(\underline{u},\overline{v}) = (\epsilon\varphi_1, \theta(x,\hat{\nu}))$. Then $(\underline{u},\overline{v})$ satisfies

$$\hat{\mu}\Delta\underline{u} + \underline{u}[m(x) - \underline{u} - b\overline{v}] = \epsilon\varphi_1(-\lambda_1 - \epsilon\varphi_1) \geq 0 \tag{5.15}$$

for small $\epsilon > 0$, and

$$\hat{\nu}\Delta\overline{v} + \overline{v}[m(x) - c\underline{u} - \overline{v}] = -\epsilon\theta(x, \hat{\nu})\varphi_1 \leq 0. \tag{5.16}$$

Since $\hat{u} \geq \underline{u}$ and $\overline{v} \geq \hat{v}$, and they all satisfy zero flux boundary conditions, by supersolution and subsolution method [34] we see that (1.5) has a positive solution for every $b \in (\hat{b}, b_*)$ and $(\hat{\mu}, \hat{\nu}) \in \Sigma_c$. However, this contradicts the definition of b_*.

Hence, (1.5) has no positive steady state if $c \in (c_*, 1)$, $b \in (0, b_*)$ and $(\mu, \nu) \in \Sigma_c$. Since $(\theta(x, \mu), 0)$ is asymptotically stable while $(0, \theta(x, \nu))$ is unstable, from [19] we see that $(\theta(x, \mu), 0)$ is globally asymptotically stable. This completes the proof of Theorem 1.9. $\square$

Acknowledgment. The author would like to thank Prof. Thomas Nagylaki for stimulating discussions and helpful comments, and Ms. Yun Fu and Prof. Ed Overman for carrying out the numerical computations. This work is supported in part by NSF grant upon Agreement No. 0112050.

References

1. J.M. Arrieta, A.N. Carvalho, and A. Rodriguez-Bernal, Upper semicontinuity for attractors of parabolic problems with localized large diffusion and nonlinear boundary conditions, J. Diff. Eqs. **168** (2000) 33-59.
2. R. S. Cantrell and C. Cosner, The effects of spatial heterogeneity in population dynamics, J. Math. Biol. **29** (1991) 315-338.
3. R. S. Cantrell and C. Cosner, Should a park be an island? SIAM J. Appl. Math. **53** (1993) 219-252.
4. R. S. Cantrell and C. Cosner On the effects of spatial heterogeneity on the persistence of interacting species, J. Math. Biol. **37** (1998) 103-145.
5. R. S. Cantrell and C. Cosner, Spatial Ecology via Reaction-Diffusion Equations, Series in Mathematical and Computational Biology, John Wiley and Sons, Chichester, UK, 2003.
6. R. S. Cantrell, C. Cosner, and V. Hutson, Permanence in ecological systems with diffusion, Proc. Roy. Soc. Edinburgh **A123** (1993a) 553-559.
7. R. S. Cantrell, C. Cosner, and V. Hutson, Ecological models, permanence and spatial heterogeneity, Rocky Mount. J. Math. **26** (1996) 1-35.
8. R.S. Cantrell, C. Cosner, and Y. Lou, Multiple reversals of competitive dominance in ecological reserves via external habitat degradation, J. Dyn. Diff. Eqs. **16** (2004) 973-1010.
9. A.N. Carvalho and J.K. Hale, Large diffusion with dispersion, Nonl. Anal. **17** (1991) 1139-1151.
10. E. Conway, D. Hoff, and J. Smoller, Large time behavior of solutions of systems of nonlinear reaction-diffusion equations, SIAM J. Appl. Math. **35** (1978) 1-16.
11. J. Dockery, V. Hutson, K. Mischaikow, and M. Pernarowski, The evolution of slow dispersal rates: a reaction-diffusion model, J. Math. Biol. **37** (1998) 61-83.
12. Y. Du, Effects of a degeneracy in the competition model, Part I. Classical and generalized steady-state solutions, J. Diff. Eqs. **181** (2002) 92-132.

13. Y. Du, Effects of a degeneracy in the competition model, Part II. Perturbation and dynamical behavior, J. Diff. Eqs. **181** (2002) 133-164.
14. Y. Du, Realization of prescribed patterns in the competition model, J. Diff. Eqs. **193** (2003) 147-179.
15. J.E. Furter and J. López-Gómez, Diffusion-mediated permanence problem for a heterogeneous Lotka-Volterra competition model, Proc. Roy. Soc. Edin. **127A** (1997) 281-336.
16. D. Gilbarg and N. Trudinger Elliptic Partial Differential Equation of Second Order, 2nd Edition, Springer-Verlag, Berlin, 1983.
17. J.K. Hale, Large diffusivity and asymptotic behavior in parabolic systems, J. Math. Anal. Appl. **118** (1986) 455-466.
18. A. Hastings, Spatial heterogeneity and ecological models, Ecology **71** (1990) 426-428.
19. P. Hess, Periodic Parabolic Boundary Value Problems and Positivity, Longman Scientific & Technical, Harlow, UK, 1991.
20. E.E. Holmes, M.A. Lewis, J.E. Banks, and R.R. Veit, Partial differential equations in ecology: spatial interactions and population dynamics. Ecology **75** (1994), 17-29.
21. S. Hsu, H. Smith, and P. Waltman, Competitive exclusion and coexistence for competitive systems on ordered Banach spaces, Trans. Amer. Math. Soc. **348** (1996) 4083–4094.
22. V. Hutson, J. López-Gómez, K. Mischaikow, and G. Vickers, Limit behavior for a competing species problem with diffusion, in *Dynamical Systems and Applications*, World Sci. Ser. Appl. Anal. 4, World Scientific, River Edge, NJ, 1995, 501–533.
23. V. Hutson, K. Mischaikow, and P. Poláčik, The evolution of dispersal rates in a heterogeneous time-periodic environment, J. Math. Biol. **43** (2001) 501-533.
24. V. Hutson, Y. Lou, and K. Mischaikow, Spatial heterogeneity of resources versus Lotka-Volterra dynamics, J. Diff. Eqs. **185** (2002) 97-136.
25. V. Hutson, Y. Lou, and K. Mischaikow, Convergence in competition models with small diffusion coefficients, J. Diff. Eqs. **211** (2005) 135-161.
26. V. Hutson, Y. Lou, K. Mischaikow, and P. Poláčik, Competing species near the degenerate limit, SIAM J. Math. Anal. **35** (2003) 453-491.
27. V. Hutson, S. Martinez, K. Mischaikow, and G.T. Vickers, The evolution of dispersal, J. Math. Biol. **47** (2003) 483-517.
28. M. Iida, M. Tatsuya, H. Ninomiya, and E. Yanagida, Diffusion-induced extinction of a superior species in a competition system, Japan J. Iudust. Appl. Math. **15** (1998) 223-252.
29. J. López-Gómez, Coexistence and meta-coexistence for competing species, Houston J. Math. (2003) 483-536.
30. J.D. Murray, Mathematical biology. II. Spatial models and Biomedical Applications. 3rd edition. Interdisciplinary Applied Mathematics, 18. Springer-Verlag, New York, 2003.
31. H. Ninomiya, Separatrices of competition-diffusion equations, J. Math. Kyoto Univ. **35** (1995) 539-567.
32. A. Okubo and S. Levin, Diffusion and Ecological Problems: modern perspectives. Second edition. Interdisciplinary Appl. Math., **14** Springer-Verlag, New York, 2001.
33. S. Pacala and J. Roughgarden, Spatial heterogeneity and interspecific competition, Theor. Pop. Biol. **21** (1982) 92-113.
34. C.V. Pao, Nonlinear Parabolic and Elliptic Equations, Plenum, New York, 1992.
35. A.B. Potapov and M.A. Lewis, Climate and competition: the effect of moving range boundaries on habitat invasibility, Bull. Math. Biol. (2004) 975-1008.
36. M. H. Protter and H. F. Weinberger, Maximum Principles in Differential Equations, Second Edition, Springer-Verlag, Berlin, 1984.
37. N. Shigesada and K. Kawasaki, Biological Invasions: Theory and Practice, Oxford Series in Ecology and Evolution, Oxford University Press, Oxford, New York, Tokyo, 1997.