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Title: Persistence Landscapes are Graded Persistence Diagrams

The standard construction of the persistence diagram of a persistence module $M_1 \to M_2 \to \cdots \to M_n$ proceeds roughly in two steps. First, define the rank function rank by $\text{rank}_i^j = \text{rank}(M_i \to M_j)$. Second, "differentiate" the rank function via inclusion-exclusion to obtain the persistence diagram function PD. The inclusion-exclusion differentiation process is a specific example of Möbius inversion on a partially ordered set, and the function PD is the derivative of the rank function in the sense that $\text{rank}_i^j = \sum_{i' \leq i \leq j \leq j'} PD_{i'}^{j'}$.

We extend persistence diagrams by introducing a strictly richer data structure, the *graded persistence diagram*. Our construction defines the *graded* rank function, rank_*. For example, if $\text{rank}_i^j = 3$, then $(\text{rank}_*)_i^j$ is the tuple $(1, 1, 1, 0, 0, \ldots)$. Differentiating rank_* results in a *graded persistence diagram* Λ. We show that the graded persistence diagram is equivalent to a previously studied object: the module's persistence landscape. We also show the graded persistence diagram enriches the standard diagram by exhibiting that the following diagram commutes:

$$\begin{array}{ccc} \text{Rank function, rank} & \underset{\text{Summation}}{\overset{\text{Differentiation}}{\rightleftarrows}} & \text{Persistence diagram, } PD \\ \text{Grading}\downarrow \quad \uparrow\Sigma & & \uparrow\Sigma \\ \text{Graded rank, rank}_* & \underset{\text{Summation}}{\overset{\text{Differentiation}}{\rightleftarrows}} & \text{Graded persistence diagram, } \Lambda \end{array}$$